UNIVERSITY OF CAMBRIDGE

MAMA/335, NST3AS/335, MAAS/335

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 1:30 pm to 3:30 pm

PAPER 335

DIRECT AND INVERSE SCATTERING OF WAVES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the two-dimensional problem of propagation of a time-harmonic plane wave initially given by $\psi(x,z) = e^{ikx}$.

Assume that the wave is propagating in free space and incident at $x = -\xi$ onto a vertical layer in the region $x \in [-\xi, 0]$. The layer has refractive index $n(x,z) = 1 + w(z)$, where $w(z)$ is a continuous random fluctuation with mean zero, stationary in z, with variance $\sigma^2 = \langle w(z)^2 \rangle \ll 1$, and normally distributed. We assume that the field acquires only a phase change on going through the layer, which therefore acts as a 'phase screen'. Under the assumption of weak scattering, the parabolic wave equation holds, and calculations can be carried out for the reduced wave $E(x,z)$.

(i) Calculate the average of the field $\langle E(x,z) \rangle$ at $x = 0$, on emerging from the phase screen. Comment on whether this result can also hold approximately if the p.d.f. (probability distribution function) of the random refractive index fluctuations is not normally distributed.

Then derive an equation for the evolution of the mean field $\langle E(x,z) \rangle$ for $x \geqslant 0$.

(ii) By using Fourier transforms, or otherwise, derive an equation for the evolution of the second moment defined by $\langle E(x,z_1)E(x,z_2) \rangle$ for $x \geqslant 0$ and write an expression for the solution for this second moment at an arbitrary value x.

(iii) Now write the total field $E(x,z_j)$ as a sum of a coherent field $\langle E(x,z_j) \rangle$ and an incoherent (diffuse) field $E_d(x,z_j) = A_j + iB_j$, where A_j and B_j are respectively the real and imaginary parts of $E_d(x,z_j)$, and $j = 1,2$, so

$$E_j = \langle E \rangle + A_j + iB_j \ . \tag{1}$$

Using this notation, and given that we can take $\langle A_j \rangle = \langle B_j \rangle = 0$ because of the assumption of weak scatter, and disregarding terms of order greater than 2 in the diffuse field, derive an expression just in terms of A_j for the spatial correlation of intensity fluctuations, ρ_I, defined as

$$\rho_I = \frac{\langle I_1 I_2 \rangle - \langle I \rangle^2}{\langle I^2 \rangle - \langle I \rangle^2} \ , \tag{2}$$

where $I = E(x,z)E^*(x,z)$ is the intensity of the field at (x,z), I_1 and I_2 are the intensities measured at two points (x,z_1) and (x,z_2) in the same x plane, and the subscripts 1 and 2 refer to coordinates z_1 and z_2.

2 An acoustic field $\psi(\mathbf{r})$ is generated by an incident plane wave $\psi_i(\mathbf{r}) = e^{ik_0\hat{\mathbf{x}}_0\cdot\mathbf{r}}$ travelling in the direction of the unit vector $\hat{\mathbf{x}}_0$ in a 3-dimensional non-scattering medium which includes an inhomogeneity occupying a region D, so that the refractive index $n(\mathbf{r})$ is:

$$n(\mathbf{r}) = \begin{cases} 1 + n_\delta & \text{if } \mathbf{r} \in D \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

where D is a bounded, connected domain containing the origin, k_0 is real-valued, and we assume that $n_\delta(\mathbf{r}) \ll 1$. The scattered field $\psi_s = \psi - \psi_i$ satisfies the Helmholtz equation

$$\nabla^2\psi_s + k^2(\mathbf{r})\psi_s = 0 \ , \tag{2}$$

where $k(\mathbf{r}) = k_0 n(\mathbf{r})$, with Sommerfeld boundary condition at infinity.

(i) Consider the inverse problem of recovering the refractive index of the inhomogeneity D from measurements of the scattered field $\psi_s(\mathbf{r})$ in the far fiield.

Using the first order Born approximation for the scattered field, derive a far field approximation for $\psi_s(\mathbf{r})$ and relate this to the far field pattern at $\mathbf{r}$ generated by an incident plane wave in the direction $\hat{\mathbf{x}}_0$, denoted by $f_\infty(\hat{\mathbf{x}}_0, \mathbf{r})$.

Hence, write an expression giving the formal solution of this inverse problem, i.e. the so-called scattering potential $V(\mathbf{r}) = k_0^2(1 - n^2(\mathbf{r}))$.

(ii) Define the operator $T : L^2(D) \mapsto L^2(S_1)$ which maps the scattering potential $V(\mathbf{r}) \in L^2(D)$ onto a far field pattern $f_\infty(\hat{\mathbf{r}}, k) \in L^2(S_1)$, where S_1 is the unit sphere, and $\hat{\mathbf{r}}$ is the unit vector in the direction $\mathbf{r}$.

Given the usual definition of inner product for functions in Hilbert spaces, consider the Fourier transform operator $\mathcal{F}$ operating on the space of functions $g \in L^2(\mathbb{R}^3)$ and the inverse Fourier transform operator $\mathcal{F}^{-1}$, which also operates on the same space. Find the adjoint $\mathcal{F}^*$ of $\mathcal{F}$.

Hence find an expression for the adjoint T^* of T, and deduce that the solution $V(\mathbf{r})$ found for the inverse problem in (i) is the least squares solution of this inverse problem.

(iii) Comment on possible sources of ill-posedness for this inverse problem.

3 Consider the inverse problem of finding x, given data y, from

$$Ax = y \ , \tag{1}$$

where $A : X \to Y$ is a given compact linear operator between two Hilbert spaces, and $x \in X, \ y \in Y$.

(i) Give the definition of a regularisation strategy for this inverse problem, for given noisy data, i.e. when the r.h.s. of (1) is some y^δ with $\| y^{(\delta)} - y \| \leqslant \delta$

Explain why a regularisation strategy is needed to obtain solutions to this inverse problem with noisy data.

Define Tikhonov regularisation and write a formal expression for the Tikhonov regularised solution of (1).

Explain briefly why the Tikhonov regularised solution is stable.

(ii) Define a *singular value system* $\{\sigma_i; u_i, v_i\}$ for A, and show how it can be used to construct the Tikhonov regularised solution.

(iii) The iterated Tikhonov method for (1) is defined by:

$$\alpha x_{n+1} = \alpha x_n - A^* A x_n + A^* y \tag{2}$$

for some $\alpha > 0$, and with the first term x_0 given by the Tikhonov regularised solution derived in (ii). Note that, similarly to Landweber iteration, the $(n+1)^{th}$ term in iterated Tikhonov can be written in closed form, i.e. as a function of $A^* y$ only, and not of x_n.

By using the singular value system and the closed form of x_n, show that the n^{th} iterate in the iterated Tikhonov method can be written as

$$x_n = \sum_{i=1}^{\infty} g_\alpha(y, u_i) v_i \ , \tag{3}$$

where $(\cdot, \cdot)$ denotes an inner product.

Derive an expression for the function g_α.

[Hint: you may assume that an analogous expression to the partial sum of a geometric series $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ *holds for the operators appearing here.]*

END OF PAPER