MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 9:00 am to 12:00 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Shallow water equations

(i) Consider a layer of fluid of homogeneous density, ρ, with an upper free surface at $z = H_0 - h_b + \eta(x, y, t)$ and a rigid lower boundary at $z = -h_b$ and where η is the free surface height and H_0 is the mean height. State the shallow water (SW) equations for this system. What assumptions are required to derive these equations? Briefly describe why the SW system is useful in the study of the atmosphere and the ocean.

Derive an expression for the shallow water potential vorticity, showing that it is materially conserved.

(ii) Consider the above system with $h_b = 0$ on an f-plane. In the initial state, the free surface is flat, so that the fluid layer everywhere has depth H_0 and the velocity components are

$$u(x, y, 0) = 0, \quad v(x, y, 0) = v_0 \, \mathrm{sgn}(x), \tag{1}$$

where v_0 is a constant. (Note: $\mathrm{sgn}(x) = x/|x|$ for $x \neq 0$ and 0 for $x = 0$.) Assume that the SW equations apply and may be linearised about the state of rest with uniform depth H_0. Show, carefully justifying your reasoning, that in the ultimate steady state the free-surface displacement $\eta(x, y)$ is given by

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{R_d^2}\right] \eta = F(x), \tag{2}$$

where you should give expressions for R_d and $F(x)$. Hence derive the steady-state solutions for η, u and v.

Calculate the total potential and kinetic energies per unit length in the y-direction, in the region $-L < x < L$, with $L \gg R_d$, at early times and in the steady state. Comment on the mechanism by which this system adjusts to steady state explaining why the total energy in this region changes. Give an estimate, in terms of f, g, H_0 and L, of the time required for the energy in this region to reach its steady state value.

2 Ocean circulation

(i) Sverdrup balance can be written

$$\beta v = W(x, y), \tag{1}$$

where $\beta = df/dy$ is the north/south gradient of the Coriolis acceleration, v is the velocity in the northwards direction and $W(x, y)$ is the curl of the surface wind stress. Describe the assumptions required for Sverdrup balance to hold. Briefly ($\sim$ 1-2 sentences) provide a physical interpretation for Sverdrup balance.

(ii) Consider a rectangular ocean basin with boundaries at $x = 0$, $x = L_X$, $y = 0$ and $y = L_Y$ and the following wind stress curl,

$$W(y) = -\sin(2\pi y/L_Y). \tag{2}$$

Find the streamfunction, ψ, corresponding to Sverdrup flow in the interior of the ocean basin. Sketch streamlines associated with the flow in Sverdrup balance, indicating the direction of the circulation with arrows. In the presence of viscosity and applying the no-slip boundary condition, there are viscous boundary layers at each boundary. For a constant kinematic viscosity, ν, estimate the thickness of the viscous boundary layers on all four horizontal boundaries of the ocean basin, using a scaling analysis or otherwise.

(iii) In the presence of a *constant* surface wind stress curl, $W_0 > 0$, the shallow water potential vorticity equation is

$$\frac{D}{Dt}\left(\frac{\zeta + f}{H}\right) = \frac{W_0}{H}, \tag{3}$$

where ζ is the relative vorticity, H is the fluid depth, and D/Dt is the material derivative. Consider a case where

$$H = y^2, \tag{4}$$

in the domain $0 \leqslant x \leqslant L_X$, $-L_Y \leqslant y \leqslant L_Y$ and with constant $f = f_0$. Assuming that the Rossby number is small, obtain an expression for the streamfunction, ψ, associated with steady flow and sketch the streamlines, indicating the direction of circulation using arrows. Clearly state any additional assumptions that you make.

(iv) Consider water parcels that follow the streamlines that you found in part (iii) for $y > 0$ and $y < 0$. Briefly ($\sim$ 3-4 sentences) discuss the contributions to the potential vorticity along these two paths.

3 Mountain waves

(i) The quasi-geostrophic potential vorticity (QGPV) equation can be written

$$q_t + \psi_x q_y - \psi_y q_x = 0, \tag{1}$$

where subscripts denote partial derivatives, and the PV is given by

$$q = \psi_{xx} + \psi_{yy} + \frac{f_0^2}{N^2}\psi_{zz} + \beta y, \tag{2}$$

where ψ is the streamfunction, $f = f_0 + \beta y$ is the Coriolis parameter, and N is the (constant) background buoyancy frequency. State the assumptions that are required for the QGPV equations to hold.

(ii) Consider a rotating stratified atmosphere that satisfies the QGPV equation and is bounded from below by a flat rigid boundary located at $z = 0$. Linearising about a state of rest, seek solutions for the perturbation streamfunction of the form

$$\psi' = \Re\left(\widehat{\psi}(z)e^{i(kx+ly-\omega t)}\right), \tag{3}$$

where $\Re(A)$ denotes the real part of A. Write the no normal flow boundary condition at $z = 0$ in terms of deriviatives of the perturbation streamfunction, ψ', and find the dispersion relation for small amplitude waves. Provide a brief interpretation for waves with large and small horizontal wavelength (‘long’ and ‘short’ waves, respectively) and identify the required condition for each of these limits to be valid.

(iii) Consider a basic state with a velocity $\mathbf{u} = U\hat{\mathbf{x}}$ flowing over a sinusoidal mountain range where the height of the ground is $h = h_0\cos(kx)$ and U, h_0, and k are constant. Find the dispersion relation for small amplitude waves generated by flow over the mountains that satisfy the QGPV equation as written in part (i), where the perturbation streamfunction has the form

$$\psi' = \Re\left(\widehat{\psi}(z)e^{i(kx+ly)}\right). \tag{4}$$

Identify a condition that must be satisfied for the waves to propagate vertically.

(iv) For the same flow as described in part (iii), write the no normal flow boundary condition on the perturbation streamfunction, ψ', that is satisfied at $z = h$. In the case where $U > \beta/(k^2 + l^2)$ and the height of the mountain range is small enough to apply the boundary condition at $z = 0$, find an expression for $\widehat{\psi}(z)$. Briefly (1-2 sentences) comment on the influence of the buoyancy frequency, N, on the amplitude and characteristic vertical lengthscale of the waves.

4 Wave mean-flow interaction/Equatorial waves

(i) Consider the x-averaged flow on the equatorial β-plane, with x-averages denoted by $\overline{(.)}$ and with 'eddy' terms, i.e. the departure from the x-average, denoted by $'$. The equations for mean x-momentum, mass continuity of the meridional circulation and the mean buoyancy, assuming a constant background buoyancy frequency N are respectively

$$\overline{u}_t - \beta y \overline{v} = -\frac{\partial}{\partial y}(\overline{u'v'}) - \frac{\partial}{\partial z}(\overline{u'w'}), \tag{1}$$

$$\frac{\partial \overline{v}_a}{\partial y} + \frac{\partial \overline{w}_a}{\partial z} = 0, \tag{2}$$

$$\overline{\sigma}_t + N^2 \overline{w}_a = -\frac{\partial}{\partial y}(\overline{v'\sigma'}), \tag{3}$$

with the terms on the right-hand side representing the effect of eddy forcing. (A second term on the right-hand side of (3) has purposely been neglected.) $\overline{u}$ is the x-component of the mean flow. $(\overline{v}_a, \overline{w}_a)$ are the y- and z- components of the mean meridional circulation and $\overline{\sigma}$ is the mean buoyancy.

Show that a new meridional circulation $(\overline{v}_a^*, \overline{w}_a^*)$, called the transformed Eulerian mean circulation, may be defined so that all the eddy forcing terms appear on the right-hand side of (1), in the form $(\partial/\partial y)F^{(y)} + (\partial/\partial z)F^{(z)}$, with none on the right-hand side of (3). Show explicitly the form of $F^{(y)}$ and $F^{(z)}$.

(ii) Assume that the eddy terms are governed by the equations for the motion of a density-stratified fluid on an equatorial β-plane, under the Boussinesq and hydrostatic approximations, and linearised about a state of rest:

$$u'_t - \beta y v' = -\phi'_x, \tag{4}$$

$$v'_t + \beta y u' = -\phi'_y, \tag{5}$$

$$u'_x + v'_y + w'_z = 0, \tag{6}$$

$$\sigma'_t + N^2 w' = 0, \tag{7}$$

$$\phi'_z = \sigma', \tag{8}$$

where $\mathbf{u}' = (u, v, w)$ is the 3-dimensional velocity, σ' is the buoyancy and ϕ' is the pressure perturbation divided by the constant background density. Consider plane wave solutions of the form $u' = \Re(\hat{u}(y)e^{i(kx+mz-\omega t)})$, with $\hat{v}(y)$, $\hat{w}(y)$, $\hat{\sigma}(y)$ and $\hat{\phi}(y)$ defined similarly with respect to v', w', σ' and ϕ'. All of these functions of y tend to zero as $|y| \to \infty$. k, m and ω are all assumed real, with $\omega > 0$. [You may assume, without loss of generality, that when it is non-zero the function $\hat{v}(y)$ is real.]

Show that these equations admit a Kelvin-wave solution with $\hat{v}(y) = 0$ and that the dispersion relation for this wave is $\omega = Nk/|m|$. Include an explanation of why $\omega = -Nk/|m|$ is not an acceptable form of the dispersion relation.

[QUESTION CONTINUES ON THE NEXT PAGE]

Use the relation between $\hat{u}$ and $\hat{\phi}$ implied by the equations to show that for Kelvin waves

$$\overline{u'w'} = -\tfrac{1}{2}\frac{km}{N^2}|\hat{\phi}|^2.$$

If upward propagating Kelvin waves are dissipated what is the sign of the force on the mean flow?

(iii) Show that all equatorial waves have $F^{(y)} = 0$ and $F^{(z)} \neq 0$ with

$$F^{(z)} = \tfrac{1}{2}\frac{km}{N^2}|\hat{\phi}|^2. \tag{9}$$

[Hint: You do not need to find explicit solutions for functions such as $\hat{v}(y)$, but you should state clearly any properties of such functions that you use.]

It may be assumed that, for all equatorial waves, upward group propagation implies $m < 0$. Comment on the implications of (9) for the sign of the force exerted on the mean flow by different types of equatorial waves that are generated in the troposphere, propagate upwards and are dissipated in the stratosphere.

Comment also on the relation of (9) to the physical interpretation of $F^{(z)}$.

END OF PAPER