MAT3

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2025 1:30 pm to 4:30 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

You should attempt **ALL** questions.
There are **FOUR** questions in total.
Questions 1 and 2 carry half the credit of questions 3 and 4.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An ice sheet of uniform temperature $T = T_{-\infty} < T_m$ calves (breaks off) into the ocean to form a large iceberg. The ocean has uniform temperature $T = T_m = 0°\mathrm{C}$ and salinity $C = C_0$. It can be assumed to have a linear liquidus temperature $T_L = T_m - mC$, where m is constant.

Draw a sketch of the one-dimensional temperature and salinity fields in the ice and the ocean near a planar surface of the iceberg, assuming that heat and mass transfer occur solely by conduction/diffusion. Write down the equations and boundary conditions governing the temperature and salinity fields in the iceberg and in the ocean, writing the position of the ice–ocean interface as $x = a(t)$, where x is the direction normal to the interface, pointing into the ocean. *You may assume that the ice and ocean have equal thermodynamic properties.*

Explain why the position of the interface takes the form $a(t) = 2\lambda\sqrt{Dt}$, where D is the diffusivity of salt. Write down (or derive if necessary) expressions for the temperature and salinity fields. Use the interfacial conditions to determine a pair of equations to determine the temperature T_i of the ice–ocean interface and the scale factor λ. [*You may like to use the function* $F(z) \equiv \sqrt{\pi} z e^{z^2} \operatorname{erfc} z$.]

Taking the limit $\epsilon \equiv \sqrt{D/\kappa} \ll 1$, where D and κ are the diffusivities for salt and heat, respectively, derive leading-order expressions for T_i and $F(\lambda)$ in terms of T_m, $T_{-\infty}$, m and C_0. Under what conditions does ice grow ($\dot{a} > 0$) or ablate ($\dot{a} < 0$)? Sketch the trajectory of (C, T) in the phase diagram in each case, indicating when constitutional supercooling might occur. What mechanism causes ablation in the latter case?

Note that

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2}\, dy \qquad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z).$$

2 Consider the injection of CO_2 of density $\rho - \Delta\rho$ at volumetric rate Q into a deep, horizontal, saline aquifer of porosity ϕ and permeability k occupying $0 < z < H$ and filled with a fluid of density ρ, where z measures distance vertically downwards. The aquifer is bounded from above by a layer of impermeable rock of thickness b, which has a fracture at horizontal position $x = L$, where x measures the distance from the point of injection. The injected CO_2 forms a long, thin gravity current of thickness $h(x,t) \ll H$ which flows along the horizon $z = 0$ in the positive x-direction as a function of time t. At the fracture, some CO_2 leaks into the overlying strata at volumetric leakage rate

$$Q_l = -W\frac{\alpha k}{\mu}\left[\frac{p(L,0,t) - p(L,-b)}{b}\right],$$

where $W \ll h$ is the width of the fracture, αk is the permeability of the fracture, $p(L,0,t)$ is the pressure in the CO_2 at the base ($z = 0$) of the fracture, $p(L,-b) \simeq p_H - \rho g H$ is the pressure above the impermeable layer and p_H is a constant reference pressure deep in the aquifer at depth $z = H$. Some CO_2 continues to leak past the fracture, spreading to distance $x_N(t) > L$.

Draw a diagram of the aquifer and the spreading CO_2, taking care to label the dimensions and physical properties of the aquifer, CO_2 current and fracture.

The current of CO_2 is long and thin, and the pressure within the current is therefore hydrostatic. Briefly derive a model for the gravity current of CO_2, paying particular attention to the flux conditions at the fracture.

At late times, when $x_N \gg L$, a steady state balance is achieved for $0 < x < L$ in which the flux Q into the aquifer is approximately equal to the leakage flux Q_l. Find an expression for the height of the steady current written solely in terms of $h_0 = h(x = 0)$, $h_L = h(x = L)$, and x/L, and find expressions for h_L and h_0 in terms of the external parameters of the system.

Finally, consider the spreading of the current into the far field ($x > L$) from the fracture, where $h_L \approx$ constant. Use a scaling analysis to show how the extent of the current, x_N, depends on time and formulate a self-similar problem which would be solved to completely describe $x_N(t)$. You do not need to solve this self-similar problem.

3 A grounded ice sheet of density ρ, treated as a Newtonian fluid of dynamic viscosity μ, slides on a horizontal layer of till, which lubricates the sheet sufficiently that resistance to internal deformation derives dominantly from extensional stresses, with respect to which internal shear stresses are negligible. Such a so-called *shelfy stream* acts like a floating ice shelf except that there is a basal stress

$$\tau_b = \mu \frac{u}{\lambda}$$

resisting the motion, where u is the vertically uniform, quasi-horizontal velocity of the ice sheet and λ is a physical parameter characterising the till. Draw a sketch of the physical setup. How might λ be related to the thickness of the layer of till and its viscosity? What are its dimensions?

By considering force balances on a section of the ice sheet between x and $x + \delta x$, where x measures downstream distance from the grounding line, derive the momentum equation

$$4\mu \frac{\partial}{\partial x}\left(h \frac{\partial u}{\partial x}\right) - \rho g h \frac{\partial h}{\partial x} - \mu \frac{u}{\lambda} = 0,$$

where h is the thickness of the ice, giving careful explanations of each stage of your derivation. Write down the corresponding equation describing an ice shelf floating on an ocean of density ρ_w, into which the shelfy stream flows across a grounding line located at $x = 0$.

Similarly, derive an equation describing local conservation of mass, integrated through the depth of the ice sheet or shelf.

Show that, in regions where thin-film theory is valid, the extensional stresses in the stream are negligible in comparison with the basal friction provided $\lambda \ll h_G$, where h_G is the thickness of the ice at the grounding line. In such a case, with constant volumetric flux per unit width q, and with the base of the stream a uniform depth b below sea level, find the steady state ice thickness in the form

$$h = h_G \left(1 - \frac{x}{\delta}\right)^{\alpha},$$

where h_G, δ and α should be determined/defined in terms of the physical parameters given above. By considering the surface slope of the stream at the grounding line, find a condition on λ that ensures the validity of thin-film theory for the solution just found. Show that both conditions on λ can be met provided that

$$\frac{\mu q}{\rho g h_G^3} \ll 1.$$

From the equations and boundary conditions governing the evolution of the shelf, show that the depth-integrated horizontal force exerted by the shelf on the stream is equal to the hydrostatic force of the ocean at the grounding line. Assuming that the full horizontal force in the shelfy stream (including extensional stresses) can be determined to leading order from the approximate solution just found, determine a relationship between q and h_G.

[*Note, for information only, that if q and h_G are slowly varying with x then the solution found above can be used to relate conditions at the grounding line, and therefore $q(h_G)$ can be used to determine the position of the grounding line.*]

4 Observations of a floating ice sheet depict a series of corrugations on the underside of the ice shelf of wavelength $\lambda \ll H$ which form at the grounding zone, where H is the thickness of the floating ice. Consider the relaxation of the corrugations as a function of time. You should neglect the advection of the ice shelf away from the grounding line, and any large-scale internal thinning, instead focusing on the evolution of the corrugations.

To do so, consider a two-dimensional model of the viscous relaxation of the ice. You should treat the ice as having Newtonian viscosity μ and density ρ, floating on water of negligible viscosity and density ρ_w. The corrugations have the complex representation

$$z = \eta(x,t) = \hat{\eta} e^{ikx+\sigma t},$$

with amplitude $\hat{\eta} \ll \lambda = 2\pi/k$, where k is the wavenumber and σ is the decay rate of the corrugations in time t.

Draw a sketch of the problem, taking care to label the corrugations, ice thickness and coordinate axes. Inertia of the ice is negligible, so flow is determined by the Stokes equation. You should then express the velocity within the ice in terms of a streamfunction, $\mathbf{u} = \nabla \times \psi \hat{\mathbf{y}}$. Take the curl of the Stokes equation and show that, by applying appropriate far field boundary conditions, the streamfunction is given by

$$\psi = (A + Bz) e^{-kz} e^{ikx+\sigma t},$$

for constants A and B. Solve for the velocity components and pressure.

At the corrugated ice-ocean boundary, the ocean exerts zero shear stress on the ice, and the normal stress is hydrostatic. Since the perturbations to the interface are small, linearise these two boundary conditions about the unperturbed interface ($z = 0$) and solve for the constants A and B. Finally, write down a kinematic condition for the evolution of η, and so determine the growth or decay rate of corrugations.

Express your answer as the timescale $\tau(\lambda)$ for corrugations of wavelength λ to decay, and interpret your results physically. What do you expect observations to show far from the grounding line when the ice has been advected down the length of the ice shelf?

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