UNIVERSITY OF CAMBRIDGE

MAMA/329, NST3AS/329, MAAS/329

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 1:30 pm to 4:30 pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Explaining your choice of trial harmonic potentials, determine the velocity field due to a rigid sphere of radius a moving with velocity $\mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{x}$ through unbounded fluid of viscosity μ that is otherwise at rest. Determine also the stress field $\boldsymbol{\sigma}$ when $\mathbf{U} = \mathbf{0}$.

[*You may assume below that* $\boldsymbol{\sigma} = -\dfrac{3\mu}{2a^2}\big\{\mathbf{Ux} + \mathbf{xU} + (\mathbf{U} \cdot \mathbf{x})(\mathbf{I} - 2\mathbf{xx}/a^2)\big\}$ *on* $r = a$ *when* $\boldsymbol{\Omega} = \mathbf{0}$.]

(b) A force-free couple-free spherical micro-organism of radius a swims through fluid of the same density by using a surface layer of tiny flagella to prescribe a relative velocity $\mathbf{u}_s(\mathbf{x})$ between the fluid just outside the organism and the rigid body of the organism. Hence if the velocity of the organism is $\mathbf{V} + \boldsymbol{\omega} \wedge \mathbf{x}$, with $\mathbf{V}$ and $\boldsymbol{\omega}$ constants, then the fluid velocity immediately outside the organism is $\mathbf{V} + \boldsymbol{\omega} \wedge \mathbf{x} + \mathbf{u}_s(\mathbf{x})$.

State the reciprocal theorem for Stokes flow for the case of zero body force. Use the theorem to determine $\mathbf{V}$ and $\boldsymbol{\omega}$ as integrals of $\mathbf{u}_s(\mathbf{x})$. Evaluate $\mathbf{V}$ and $\boldsymbol{\omega}$ for the case

$$\mathbf{u}_s(\mathbf{x}) = (\mathbf{A} \wedge \mathbf{x}) \wedge \mathbf{x}/a^2 + (\mathbf{B} \cdot \mathbf{x})^2\, \mathbf{B} \wedge \mathbf{x}/a^3\,, \qquad (*)$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors (true and pseudo, respectively).

[*You may use* $\displaystyle\int_{r=a} x_i x_j x_k x_l \, dS = \frac{4\pi a^6}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$.]

(c) Suppose that $\mathbf{B} = \mathbf{0}$ in $(*)$. Using some of your answer to part (a), or otherwise, and noting the force-free condition on the micro-organism, write down the form of the external velocity field $\mathbf{u}(\mathbf{x})$ in $r > a$. Verify that, with a suitable choice of the multiplicative constant, it satisfies the kinematic boundary condition.

Suppose, instead, that $\mathbf{A} = \mathbf{0}$ in $(*)$. Write down the form of a harmonic potential for the external velocity field. [You do not need to find the multiplicative constant.] How rapidly does the external velocity decay as r increases?

2 (a) An axisymmetric thread of viscous fluid falls vertically from a nozzle through air of negligible viscosity and constant pressure p_e. The thread has cross-sectional area $A(z,t) = \pi a^2$, where $|\partial a/\partial z| \ll 1$. Inertia and surface tension are negligible. Derive the equations governing the cross-sectional area A and the vertical velocity w, explaining your argument carefully.

Consider steady flow with constant flux. Let $W = w/\hat{w}$ and $Z = z/\hat{z}$, where $\hat{w}$ is a given velocity scale and $\hat{z}$ is a vertical scale to be chosen. Find the choice of $\hat{z}$ that makes the momentum equation as simple as possible. Using an integrating factor, or otherwise, show that $W(Z)$ then satisfies the first-order equation

$$\tfrac{1}{2}W'^2 = CW^2 + W\,,$$

where C is a constant of integration.

Solve this equation subject to $W \to 0$ as $Z \to 0_+$ for each of the cases (i) $C = 0$, (ii) $C = \frac{1}{2}$, (iii) $C = -\frac{1}{2}$. The nozzle is at $Z = Z_0$, where Z_0 is a small positive number. Sketch the form of $A(Z)$ for $Z > Z_0$ for each of the three cases, and describe the physical difference between the three situations in terms of the vertical stress.

(b) To manufacture very thin glass sheets, a long thin ribbon of molten glass is stretched by passing it between two pairs of rollers, with the second pair being rotated faster than the first so that the ribbon must speed up and thin by mass conservation. Model this as follows:

Let the rollers be at $z = 0$ and $z = L$, and assume that the ribbon has a rectangular cross-section $-h \leqslant x \leqslant h$, $-b \leqslant y \leqslant b$, where $h(z) \ll b(z) \ll L$. Inertia and gravity are negligible, but surface tension γ acts at the edges $\pm b$ of the sheet, rounding them into approximately semi-circular shapes of radius h. As in part (a), the fluid has constant viscosity, the surroundings have negligible viscosity and constant pressure p_e, and the axial velocity $w(z)$ is uniform in a cross-section.

Use the dynamic boundary conditions and lateral force balance to show that

$$\sigma_{zz} = -p_e - \frac{\gamma}{2h} + 3\mu\frac{dw}{dz}\,.$$

[*Hint: σ_{xx} and σ_{yy} are each uniform, but different.*] Show also from the kinematic boundary condition that

$$w\frac{db}{dz} + \frac{b}{2}\frac{dw}{dz} + \frac{\gamma b}{4\mu h} = 0\,. \qquad (1)$$

You are given that in steady flow

$$hbw = Q\,, \qquad 3\mu hb\frac{dw}{dz} + \frac{\gamma b}{2} = F\,. \qquad (2,3)$$

Describe the physical meaning of these equations. Deduce from (1)–(3) that w and b obey

$$\frac{b'}{b} + \frac{1}{2}\frac{w'}{w} = -\frac{\Gamma b}{4}, \qquad \frac{3w'}{w} = -\frac{\Gamma b}{2} + T\,,$$

where the constants Γ and T should be determined. What are their dimensions? Show that $w \propto b\exp(Tz/2)$ and solve the remaining equation for $1/b$ subject to $b(0) = b_0$.

For the case $\Gamma = 0$, find the value of F that results in a prescribed amount of thinning $h(L)/h(0)$.

3 A rigid cylinder of radius a falls through viscous fluid with its axis horizontal and parallel to a rigid vertical wall. With respect to suitably chosen coordinates (x, y), the wall lies along $y = 0$ and has velocity $(-U, 0)$, the axis of the cylinder is at $\big(0, (1 + \frac{1}{2}\epsilon)a\big)$ and is stationary, the cylinder rotates about its axis with angular velocity Ω, and the fluid occupies the region outside the cylinder in $y > 0$. Assume throughout that $0 < \epsilon \ll 1$.

Use the lubrication approximation to determine the flow in the thin gap between the cylinder and the wall. Show that the flux q through the gap is $\frac{1}{3}\epsilon a(\Omega a - U)$ (per unit axial length of cylinder).

$\Big[$*You may assume that if* $I_n \equiv \displaystyle\int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^n}$ *then* $I_1 = \pi$, $I_2 = \dfrac{\pi}{2}$ *and* $I_3 = \dfrac{3\pi}{8}$.$\Big]$

Show also that

$$\left.\frac{\sigma_{xy}}{\mu}\right|_{y=0} = \frac{4U - 2\Omega a}{h} + \frac{6q}{h^2},$$

where $h(x)$ is the gap width, and find a similar expression for σ_{xy} on $y = h$. Hence calculate the tangential force (per unit length of cylinder) that is exerted by the shear stress in the thin gap (i) on the wall and (ii) on the cylinder. Why are these forces not equal and opposite?

The cylinder has a uniform density $\Delta\rho$ greater than that of the fluid. What is the force and couple on the cylinder? Use your answers to (i) and (ii) to find U and to show that $\Omega = 0$ at leading order.

For $\Omega = 0$, sketch, as functions of x, the pressure $p(x)$ and the shear stress $\sigma_{xy}(x, h)$ on the cylinder. [*You need not find p explicitly.*] Identify the two points where the shear stress vanishes on the cylinder. Sketch the streamlines of the flow in the thin gap.

Now consider a two-dimensional Couette flow with rigid walls at $y = \pm(1 + \frac{1}{2}\epsilon)a$ which have velocity $(\pm U, 0)$. A force-free couple-free rigid cylinder of radius a is introduced perpendicular to the flow with its axis initially located at $(0, \frac{1}{2}\lambda\epsilon a)$, where $-1 < \lambda < 1$. Adapt your previous results to deduce the approximate speed V of the cylinder. For $\lambda = 0$ sketch the streamlines on either side of the cylinder, and explain why its angular velocity is $O(\epsilon^{1/2}U/a)$.

END OF PAPER