UNIVERSITY OF CAMBRIDGE

MAMA/327, NST3AS/327, MAAS/327

MAT3

MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 9:00 am to 11:00 am

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove the Paley-Wiener-Schwartz theorem.

(b) Let $P : \mathbf{C} \to \mathbf{C}$ be a non-constant polynomial. Consider the differential equation

$$P(D)u = v \in \mathcal{S}'(\mathbf{R}).$$

(i) Show that if v has compact support, then there exists a solution $u \in \mathcal{E}'(\mathbf{R})$ if and only if $z \mapsto \hat{v}(z)/P(z)$ is an entire function.

(ii) If v has compact support but $z \mapsto \hat{v}(z)/P(z)$ is *not* entire, does this preclude the existence of a solution $u \in \mathcal{S}'(\mathbf{R})$? Justify your answer.

2

(a) Let $X \subset \mathbf{R}^n$ be open. Define the *Sobolev space* $H^s(\mathbf{R}^n)$ and the *local Sobolev space* $H^s_{\text{loc}}(X)$. For polynomial P, what does it mean to say the operator $P(D)$ is *elliptic*?

(b) State, without justification, which of the following are true and which are false:

(i) If $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $u \in H^s(\mathbf{R}^n)$ then $\varphi u \in H^s(\mathbf{R}^n)$.

(ii) If $s > t$ then $H^s(\mathbf{R}^n) \subset H^t(\mathbf{R}^n)$.

(iii) If $P(D)$ is elliptic then $P(\lambda) \neq 0$ on $\mathbf{R}^n \setminus \{0\}$.

(iv) $P(D)[fg] = \sum_\alpha \frac{1}{\alpha!}[D^\alpha f][P^{(\alpha)}(D)g]$.

(v) If $u \in H^s(\mathbf{R}^n)$ and $s > n/2$ then $u \in C(\mathbf{R}^n)$.

(vi) If $u \in \mathcal{S}'(\mathbf{R}^n)$ then $u \in H^t(\mathbf{R}^n)$ for some $t \in \mathbf{R}$.

(vii) If $u \in \mathcal{E}'(\mathbf{R}^n)$ then $u \in H^t(\mathbf{R}^n)$ for some $t \in \mathbf{R}$.

(viii) If $u \in H^s(\mathbf{R}^n)$ then $D^\alpha u \in H^{s-|\alpha|}(\mathbf{R}^n)$.

Here $P^{(\alpha)}(\lambda) \equiv (D^\alpha P)(\lambda)$.

(c) Let Q be an Nth order polynomial in n variables. Suppose that there exists a $\delta \in (0, 1)$ such that $|Q^{(\alpha)}(\lambda)| \lesssim |\lambda|^{-\delta|\alpha|}|Q(\lambda)|$ for all multi-indices α when $|\lambda|$ is sufficiently large. Show that

$$Q(D)u \in H^s_{\text{loc}}(X) \Rightarrow u \in H^{s+\delta N}_{\text{loc}}(X).$$

Deduce that $Q(D)$ is *hypoelliptic*, i.e. $Q(D)u \in C^\infty(X) \Rightarrow u \in C^\infty(X)$.

[*Note: throughout (c) you may use any facts stated in part (b)*]

(d) Give an example of a partial differential operator which is hypoelliptic but not elliptic. Justify your answer.

3

(a) Let $X \subset \mathbf{R}^n$ be open. What does it mean for $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ to be a *phase function*? Define the space of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$. Show that:

(i) If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbf{R}^k; N - |\beta|)$.

(ii) If $a_i \in \mathrm{Sym}(X, \mathbf{R}^k; N_i)$ for $i = 1, 2$ then $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

(iii) If $b \in C^\infty(X \times \mathbf{R}^k)$ is positively homogeneous of degree M in θ for $|\theta|$ sufficiently large, then $b \in \mathrm{Sym}(X, \mathbf{R}^k; M)$.

For a phase function Φ and $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$, briefly describe how the oscillatory integral

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x, \theta)\, \mathrm{d}\theta$$

can be used to define a linear map from $\mathcal{D}(X)$ to $\mathbf{C}$. You may assume the resulting linear map belongs to $\mathcal{D}'(X)$.

(b) Define the *singular support* of an element of $\mathcal{D}'(X)$.

(i) Show that $\operatorname{sing\,supp} I_\Phi(a) \subset \{x \in X : \nabla_\theta \Phi(x, \theta) = 0, \text{ for some } \theta \in (\mathbf{R}^k \setminus \{0\})\}$.

(ii) Define $Z(a) = \{x \in X : a(x, \theta) = 0 \text{ for all } \theta \in \mathbf{R}^k\}$. Is it true that

$$Z(a) \cap \operatorname{sing\,supp} I_\Phi(a) = \emptyset.$$

Give a proof or counterexample.

END OF PAPER