UNIVERSITY OF CAMBRIDGE

MAMA/325, NST3AS/325, MAAS/325

MAT3

MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 9:00 am to 11:00 am

PAPER 325

QUANTUM INFORMATION, FOUNDATIONS AND GRAVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Explain carefully what is meant by the reduced density matrix of a subsystem of a bipartite quantum system.

(b) Give a model of a N-dimensional quantum system S undergoing a unitary interaction with an apparatus A designed to measure an observable B of S, where B has N distinct eigenvalues $\{\lambda_i\}_{i=1}^{N}$ with corresponding eigenvectors $\{|e_i\rangle\}$ and the apparatus Hilbert space includes "pointer states" $|i\rangle$ corresponding to the observation of λ_i. If the system is initially in state $|\phi\rangle = \sum_i a_i |e_i\rangle$, what is its reduced density matrix after the interaction?

Consider the observable $P_{ij} = |\psi_{ij}\rangle\langle\psi_{ij}|$, where $|\psi_{ij}\rangle = \frac{1}{\sqrt{2}}(|e_i\rangle + |e_j\rangle)$, for some i, j with $i \neq j$. What is the probability of obtaining outcome 1 if P_{ij} is measured on the system before the interaction with the apparatus? What is the probability of obtaining outcome 1 if P_{ij} is measured on the system after the interaction with the apparatus? Comment on what this calculation explains, and on a fundamental feature of quantum theory that it does not explain.

(c) State the Einstein-Podolsky-Rosen (EPR) criterion for an element of physical reality. By considering measurements on the state of three entangled spin-$\frac{1}{2}$ particles

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)\,,$$

show that the EPR criterion contradicts the predictions of quantum theory.

2 Consider the following hypothetical devices that could be realised by physics beyond quantum theory:

(a) A device comprising two separated boxes, labelled A and B, each of which accepts a classical input bit, 0 or 1. Each box, on receiving a valid input, instantly outputs a classical bit. The outputs depend probabilistically on the inputs, with the following conditional probabilities:

$$p(00|00) = p(11|00) = \frac{1}{2}\,, \qquad p(00|01) = p(11|01) = \frac{1}{2}\,,$$
$$p(00|10) = p(11|10) = \frac{1}{2}\,, \qquad p(10|11) = p(01|11) = \frac{1}{2}\,.$$

(b) A device comprising two separated boxes, labelled A and B, each of which accepts a classical input bit, 0 or 1. Each box, on receiving a valid input, instantly outputs a qubit. The qubit outputs depend probabilistically on the inputs, with the following conditional probabilities:

$$p(\left|+\right\rangle\left|+\right\rangle|00) = p(\left|-\right\rangle\left|-\right\rangle|00) = \frac{1}{2}\,, \qquad p(\left|+\right\rangle\left|+\right\rangle|01) = p(\left|-\right\rangle\left|-\right\rangle|01) = \frac{1}{2}\,,$$
$$p(\left|+\right\rangle\left|+\right\rangle|10) = p(\left|-\right\rangle\left|-\right\rangle|10) = \frac{1}{2}\,, \qquad p(\left|0\right\rangle\left|0\right\rangle|11) = p(\left|1\right\rangle\left|1\right\rangle|11) = \frac{1}{2}\,.$$

Here $\left|0\right\rangle, \left|1\right\rangle$ are orthonormal and $\left|\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle \pm \left|1\right\rangle)$.

(c) A device comprising two separated boxes, labelled A and B, each of which accepts as input a bit, 0 or 1, together with any pure state qubit $\left|\psi_X\right\rangle$ (where $X = A$ or B). Each box, on receiving a valid input, instantly outputs a classical description of a qubit, which is either an infinite precision classical description of $\left|\psi_X\right\rangle$, which we write as $c(\psi_X)$, or an infinite precision classical description of the orthogonal qubit, $c(\psi_X^{\perp})$. These descriptions involve specified orthonormal basis states $\left|0\right\rangle$ and $\left|1\right\rangle$ and use the phase convention that $\left|\psi_X\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$ with α real, and with $\beta = 1$ if $\alpha = 0$; similarly $\left|\psi_X^{\perp}\right\rangle = \gamma\left|0\right\rangle + \delta\left|1\right\rangle$ with γ real and with $\delta = 1$ if $\gamma = 0$. For the purpose of this question we assume that it is physically possible to produce infinite precision outputs instantly. The outputs depend probabilistically on the inputs, with the following conditional probabilities:

$$p(c(\psi_A)c(\psi_B)|00) = p(c(\psi_A^{\perp})c(\psi_B^{\perp})|00) = \frac{1}{2}\,,$$
$$p(c(\psi_A)c(\psi_B)|01) = p(c(\psi_A^{\perp})c(\psi_B^{\perp})|01) = \frac{1}{2}\,,$$
$$p(c(\psi_A)c(\psi_B)|10) = p(c(\psi_A^{\perp})c(\psi_B^{\perp})|10) = \frac{1}{2}\,,$$
$$p(c(\psi_A)c(\psi_B^{\perp})|11) = p(c(\psi_A^{\perp})c(\psi_B)|11) = \frac{1}{2}\,.$$

(In each case, (a), (b) and (c), all probabilities not explicitly mentioned are zero.)

In each case, explain whether or not the device *necessarily* allows superluminal signalling, justifying your answers carefully. In case (c), your justification should include a discussion of the possible outputs when one or both boxes are given entangled qubits as inputs.

3 Write down the Schrödinger equation for two mass m particles interacting gravitationally. You may assume here, and throughout this question, that all other forces (including external gravitational fields) are negligible. Describe how the state evolves from an initial state in which particle i is in a superposition state of the form $\frac{1}{\sqrt{3}}(\psi_{0i} + \psi_{1i} + \psi_{2i})$, where ψ_{ai} is localised around horizontal position x_{ai} for $i = 1, 2$, assuming that the ψ_{ai} remain localized around their respective horizontal positions throughout the evolution.

Suppose now that the separations $|x_{a1} - x_{a2}| = d$ are small and identical, while the other separations are sufficiently large that the gravitational potential between the masses in these locations is relatively negligible, and that the particles maintain all these relative separations throughout. By calculating the reduced density matrix of one particle, or otherwise, show that the system reaches a maximally entangled state. If $m = 10^{-14}$kg and $d = 2 \times 10^{-4}$m, estimate (to within 20%) when it reaches that state.

Does the system remain entangled for all times $t > 0$? Justify your answer.

(You may take Newton's gravitational constant $G = 6.7 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and Planck's constant $h = 6.6 \times 10^{-34}$ m^{2}kg s^{-1}, with $\hbar = h/2\pi$.)

4 Consider the following thought experiment. Alice has a particle of mass m_A that is either (a) in a superposition $|\psi\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$ or (b) an equal mixture of the states $|L\rangle$ and $|R\rangle$, which are well localized around the origin and the point $(d, 0, 0)$ respectively. Bob is in a laboratory at location $(R, 0, 0)$, where $R \gg d > 0$. His laboratory contains a test particle of mass m_B in the ground state of a very narrow harmonic trap, localized to within $\Delta X \approx l_P$, where $l_P = (\hbar G/c^3)^{1/2}$ is the Planck length. (You may assume this localization is possible and the minimum possible.) At $t = 0$ Bob either opens the trap, releasing the particle, or keeps it closed.

Describe the approximate behaviour of the joint system of Alice's and Bob's particles if the trap is opened, and give an estimate for the time after which they become entangled. You may assume that Ehrenfest's theorem applies to Bob's particle, that Alice's particle is negligibly displaced by the Newtonian gravitational interaction, and that no other forces are relevant.

Hence show that any experiment that allows Alice to distinguish between cases (a) and (b) must take at least time of order (i.e. up to numerical factors) $T_A \approx (m_A/m_P)(d/c)$, where $m_P = (\hbar c/G)^{1/2}$ is the Planck mass.

END OF PAPER