MAMA/323, NST3AS/323, MAAS/323

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 9:00 am to 11:00 am

PAPER 323

QUANTUM INFORMATION THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **Question 1** and **TWO** other questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let us consider the setting of hypothesis testing in scenarios where the hypotheses are given by quantum channels. For finite-dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$ and a linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, we define the following norms:

$$\|T\|_1 := \sup\{\|T(X)\|_1 \mid \|X\|_1 \leqslant 1\}\,,$$
$$\|T\|_\diamond := \sup_{n\in\mathbb{N}}\{\|T \otimes \mathrm{id}_n\|_1\}\,,$$

with $\mathrm{id}_n : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ the identity map.

(i) Given two quantum channels T_1, T_2, write down the problem of symmetrically discriminating between both of them, by considering the action of both channels on the same state. Define both error types, as in the case of discrimination of quantum states. [5]

(ii) Assume that T_1, T_2 are given with a priori probabilities p and $1-p$, respectively, with $p \in [0,1]$. Using the Quantum Neyman-Pearson for the optimal error in symmetric discrimination of states (or the Holevo-Helmstrom for the optimal probability of success), write an explicit expression for the optimal error in symmetric discrimination of T_1, T_2, using arbitrary ancillas (by taking the supremum over the evaluation of the combination of channels in all possible states). How would this transform if no ancilla was used? [5]

(iii) Consider the Werner-Holevo channels: $T_+, T_- : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ given by

$$T_\pm(\rho) := \frac{1}{d \pm 1}(\mathrm{Tr}[\rho]\mathbb{1} \pm \rho^T)\,,$$

where ρ^T is the transpose of ρ. Set the a priori probability for T_+ to be $p = \frac{d+1}{2d}$. Then, show that

$$\|pT_+ - (1-p)T_-\|_\diamond = 1\,,$$
$$\|pT_+ - (1-p)T_-\|_1 = \frac{1}{d}\,,$$

using that the $\diamond$ norm of the transposition map is d. Interpret this in terms of the difference between using an ancilla or not for discrimination. [10]

2 Here, we will explore separability/entanglement of quantum states. Consider a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and an arbitrary density matrix $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$.

(i) What is the definition of ρ_{AB} being separable or entangled? [4]

(ii) What does the PPT criterion state about entanglement? For which dimensions does it provide a necessary and sufficient condition? [4]

(iii) Given $d \in \mathbb{N}$, consider $z \in \{1, -1, i, -i\}^d$. Denote

$$|z\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} z_j \, |j\rangle \ .$$

Then, show that

$$\sigma := \frac{1}{4^d} \sum_{z} |z\rangle\langle z| \otimes |z^*\rangle\langle z^*| = \frac{1}{4^d d^2} \sum_{j,k,j',k'} \left(\sum_{z} z_j z_k^* z_{j'}^* z_{k'} \, |j\rangle\langle k| \otimes |j'\rangle\langle k'| \right) .$$ [3]

(iv) Consider the maximally entangled state given by

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{d}} (|11\rangle + \ldots + |dd\rangle) \, ,$$

and define the following two qudit isotropic state $\rho_p \in \mathbb{C}^{d^2 \times d^2}$:

$$\rho_p = p \left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-p) \frac{\mathbb{1}}{d^2} \, .$$

Show that $|\Phi^+\rangle\langle\Phi^+|$ can be expressed in terms of σ as

$$\left|\Phi^+\right\rangle\left\langle\Phi^+\right| = d\sigma - \frac{\mathbb{1}}{d} + \frac{1}{d} \sum_{j=1}^{d} |j\rangle\langle j| \otimes |j\rangle\langle j| \ .$$ [4]

(v) Show that ρ_p is separable for $p \leqslant \frac{1}{1+d}$ and entangled for $p > \frac{1}{1+d}$. [5]

3

In this problem, we will prove that the data-processing inequality of the Umegaki relative entropy under quantum channels is equivalent to the property of strong subadditivity of the von Neumann entropy. For $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, and $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, the following inequality always holds

$$S(\rho_B) + S(\rho_{ABC}) \leqslant S(\rho_{AB}) + S(\rho_{BC}) \,, \qquad \text{(SSA)}$$

where $\rho_X := \mathrm{Tr}_{X^c}[\rho_{ABC}]$.

(i) Given any normalised partial trace (for example $\mathrm{Tr}_A : \mathcal{S}(\mathcal{H}_{ABC}) \to \mathcal{S}(\mathcal{H}_{ABC})$ such that $\rho_{ABC} \mapsto \mathbb{1}_A/d_A \otimes \rho_{BC}$, with d_A the dimension of $\mathcal{H}_A$), which is in particular a quantum channel, assume that the data-processing inequality for the Umegaki relative entropy under such a map holds. Then, show that this implies Eq. (SSA). [5]

(ii) For the converse direction, let us show that Eq. (SSA) implies data-processing inequality for the Umegaki relative entropy under any normalised partial trace. For that, fix $\rho'_{AB}, \rho''_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, consider $\mathcal{H}_C$ to be 2-dimensional and define:

$$\rho_{ABC} = \lambda \rho'_{AB} \otimes E_C + (1-\lambda)\rho''_{AB} \otimes F_C \,,$$

with E_C and F_C orthogonal 1-dimensional projections on $\mathcal{H}_C$, and $\lambda \in [0,1]$.

(a) Write down the definition of an operator convex function in $I = [0,1]$. [3]

(b) Using that a function f is operator concave if $-f$ is operator convex, show that Eq. (SSA) for this ρ_{ABC} is equivalent to the concavity property of the conditional entropy. [3]

(c) Show that, if f is operator concave and homogeneous (i.e. $f(tX) = tf(X)$ for all $t > 0$), then for any positive matrices X and Y,

$$\left.\frac{d}{dt}\right|_{t=0} f(X+tY) := \lim_{t\to 0} \frac{1}{t}\left(f(X+tY) - f(X)\right) \geqslant f(Y) \,.$$

[4]

(d) Prove that the conditional entropy is homogeneous and use (a)-(c) to conclude the data-processing inequality for the Umegaki relative entropy under normalised partial traces. [5]

Given any quantum channel T, using its Stinespring's representation, we can show that the data-processing inequality for the Umegaki relative entropy under T can be derived from the data-processing inequality under normalised partial traces. You do not have to show this here.

4 In this problem, we will axiomatically characterize the Umegaki relative entropy. Consider a bipartite finite-dimensional Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and define $f(\cdot\|\cdot) : \mathcal{S}(\mathcal{H}_{AB}) \times \mathcal{S}(\mathcal{H}_{AB}) \to \mathbb{R}_0^+$ as a function satisfying:

a) **Continuity:** $\rho_{AB} \to f(\rho_{AB}\|\sigma_{AB})$ is continuous for any $\sigma_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$.

b) **Additivity.**

c) **Superadditivity.**

d) **Data-processing inequality.**

Additionally, given a finite-dimensional Hilbert space $\mathcal{H}$, for any pair of states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, considering its n-fold version $\mathcal{H}^{\otimes n}$, and $\{\rho'_n\}$ a sequence in $\mathcal{S}(\mathcal{H}^{\otimes n})$, we say that f is lower asymptotically semicontinuous with respect to σ if

$$\lim_{n\to\infty} \left\|\rho^{\otimes n} - \rho'_n\right\|_1 = 0$$

implies that

$$\liminf_{n\to\infty} \frac{1}{n}\Big(f(\rho'_n, \sigma^{\otimes n}) - f(\rho^{\otimes n}, \sigma^{\otimes n})\Big) \geqslant 0\,.$$

Prove the following steps:

(i) State the definition of the Umegaki relative entropy for a pair of states $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$. State the properties of additivity, superadditivity and data-processing inequality. Using any properties presented in the course for the von Neumann entropy, prove that the Umegaki relative entropy satisfies additivity and superadditivity. [7]

(ii) Show that 'bipartite' superadditivity for the Umegaki relative entropy (as seen in the course) can be generalised to 'multipartite' superadditivity, i.e. the analogous inequality with n systems instead of 2. Similarly, justify that 'multipartite' additivity is immediate from 'bipartite' additivity. [5]

(iii) Fix $\sigma \in \mathcal{S}(\mathcal{H})$ and $n \in \mathbb{N}$. For any $\rho'_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$, show that if $f : \mathcal{S}(\mathcal{H}^{\otimes n}) \times \mathcal{S}(\mathcal{H}^{\otimes n}) \to \mathbb{R}_0^+$ satisfies continuity with respect to the first input, multipartite additivity and multipartite superadditivity, then it is lower asymptotically semicontinuous. You can use that the 1-norm satisfies the data-processing inequality. [8]

The proof is concluded by showing that the properties of additivity, data-processing inequality and lower asymptotic semicontinuity imply that f is a multiple of the Umegaki relative entropy. You do not have to show this here.

END OF PAPER