MAMA/321, NST3AS/321, MAAS/321

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 9:00 am to 11:00 am

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Steady accretion discs

A convenient Newtonian approximation for a non-spinning black hole's gravitational potential in the plane $z = 0$ is

$$\Phi = -\frac{GM}{r - r_S},$$

where M is the black hole's mass, r is cylindrical radius, r_S is the Schwarzschild radius, and G is the gravitational constant.

(a) Using this potential, find an expression for $h(r)$, the specific angular momentum of a circular orbit in the plane $z = 0$ at radius r around the black hole. What is the orbital frequency $\Omega(r)$?

By determining the horizontal epicyclic frequency, find the radius of the innermost stable circular orbit.

(b) Suppose the black hole is surrounded by a circular accretion disc aligned with the plane $z = 0$. It evolves according to

$$\partial_t \Sigma + \frac{1}{2\pi r} \partial_r \mathcal{F} = 0, \qquad \mathcal{F} = -\left(\frac{dh}{dr}\right)^{-1} \partial_r \mathcal{G}, \qquad \mathcal{G} = -2\pi \bar{\nu} \Sigma r^3 \frac{d\Omega}{dr},$$

where Σ is its surface density, $\mathcal{F}$ is the radial mass flux, $\mathcal{G}$ is the viscous torque, and $\bar{\nu}$ is the mean turbulent viscosity.

(i) What should we take to be the inner radius of the disc in a steady state? Give an argument for why we might expect $\mathcal{G} = 0$ at that radius.

(ii) Suppose the disc is in steady state receiving mass at a constant rate $\dot{M}$ at its outer radius. Show that

$$\bar{\nu}\Sigma = \frac{\dot{M}}{\pi} \left(\frac{x-1}{3x-1}\right) \left[1 - A\left(x^{-1/2} - x^{-3/2}\right)\right],$$

where $x = r/r_S$ and A is a constant that you must find. Compare to the Keplerian case.

(c) Adopting the α prescription and a perfect gas equation of state, at radii much greater than r_S the vertical structure of the disc may be calculated from

$$\partial_z P = -\rho \Omega^2 z, \qquad \partial_z F = \frac{9}{4} \alpha P \Omega, \qquad \partial_z T = -\frac{3\kappa\rho}{16\sigma T^3} F, \qquad P_g = \frac{k_B \rho T}{\mu m_p},$$

where κ is the constant opacity, μ is mean molecular weight, P_g is gas pressure, P_m is magnetic pressure and $P = P_g + P_m$. All the other symbols take their usual meanings. Suppose the magnetic pressure scales as $P_m \sim \rho r \Omega c_s$, where c_s is sound speed.

(i) Show that $\beta \sim \mathrm{Ma}^{-1}$ and thus $H/r \sim \mathrm{Ma}^{-1/2}$, where H is the semi-thickness, β is the plasma beta, and Ma is the Mach number of the orbital motion.

(ii) Solve the equations of vertical structure using an order of magnitude treatment to obtain a scaling for H, and thereby show that $\bar{\nu} \propto r^a \Sigma^b$, where a and b are constants you need to find.

2 Similarity solutions for viscous accretion

Consider an accretion disc, in Keplerian rotation, extending from $r = 0$ to $r = R(t)$ subject to a mean viscosity of the form $\bar{\nu} = Ar^m\Sigma^n$, where Σ is the surface density and A, m, and n are positive constants. The disc's total mass and angular momentum are given by

$$M = 2\pi\int_0^R r\Sigma\,dr, \qquad J = 2\pi\sqrt{GM_\star}\int_0^R r^{3/2}\Sigma\,dr,$$

where $M_\star$ is the mass of the central star. In the following we seek similarity solutions that are valid at late times when the initial conditions are not important.

(a) Suppose no torque is exerted on the disc by the star. Use dimensional analysis to find the time dependence of R and M in the form of power laws. Why must the disc spread in order to accrete?

(b) Suppose instead that no mass is accreted onto the star. Find the time dependence of R and J in this case.

(c) The evolution of Σ is governed by

$$\partial_t\Sigma = 3r^{-1}\partial_r\left[r^{1/2}\partial_r\left(r^{1/2}\bar{\nu}\Sigma\right)\right].$$

(i) By adopting

$$S = \frac{\Sigma}{\Sigma_0}\left(\frac{r}{r_0}\right)^{3/2}, \quad \bar{\nu} = \bar{\nu}_0\left(\frac{r}{r_0}\right)^m\left(\frac{\Sigma}{\Sigma_0}\right)^n, \quad x = \left(\frac{r}{r_0}\right)^{1/2}, \quad \tau = \frac{3\bar{\nu}_0 t}{4r_0^2},$$

where r_0, Σ_0, and $\bar{\nu}_0$ are dimensional constants, derive the following evolution equation for S,

$$\partial_\tau S = \partial_x^2\left(x^{2m-3n-2}S^{n+1}\right).$$

(ii) Set $m = 4$ and $n = 2$. Consider the variable $\xi = x\tau^{-1/4}$ and the similarity solution $S = \tau^{-1/4}f(\xi)$, where $f(0) = 1$.

Derive and solve the nonlinear ordinary differential equation that f must satisfy. Thereby determine the functional dependence of Σ on r and t.

Show that the total mass is conserved. Hence verify that your solution is consistent with your answer to part (b)

3 'Viscous convection' in the shearing sheet

The local dynamics of a viscous Keplerian disc incorporating the latter's radial thermal structure may be represented by the Boussinesq shearing sheet. Its governing equations are:

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla P - 2\Omega\mathbf{e}_z \times \mathbf{u} - \nabla\Phi_t - N^2\theta\mathbf{e}_x + \nu\nabla^2\mathbf{u},$$
$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = u_x, \qquad \nabla\cdot\mathbf{u} = 0,$$

where $\mathbf{u}$, P, ρ_0, and θ are velocity, pressure, constant background density, and the buoyancy variable, while $\Phi_t = -\frac{3}{2}\Omega^2 x^2$ is the tidal potential. Finally, Ω is the orbital frequency of the sheet, N is the constant radial buoyancy frequency, and ν is the constant kinematic shear viscosity.

(a) Assuming the disc is orbiting a point mass, derive the expression above for the tidal potential in the shearing sheet. Make clear all the assumptions you must make.

(b) Demonstrate that $\mathbf{u} = -\frac{3}{2}\Omega x\mathbf{e}_y$, $P = P_0$, $\theta = 0$ is an equilibrium solution to the governing equations, where P_0 is a constant. Perturb this equilibrium by small axisymmetric disturbances $\mathbf{u}'$, P', and θ' and write down their linearised equations.

(c) Assume the disturbances are $\propto \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}+st)$, where $\mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z$ is a constant real wavevector and s is a potentially complex growth rate. Thereby derive the dispersion relation:
$$s^3 + 2k^2\nu s^2 + \left[\varpi^2(1+n^2) + k^4\nu^2\right]s + n^2k^2\nu\varpi^2 = 0,$$
where $n^2 = N^2/\Omega^2$ and $\varpi^2 = (k_z^2/k^2)\Omega^2$.

(d) Suppose that $N = 0$. Find an explicit expression for s. What kind of waves does the dispersion relation describe and what is the impact of viscosity on these waves?

(e) Suppose now that $\nu = 0$ and $N^2 < 0$. Derive the instability criterion $\Omega^2 + N^2 < 0$. Discuss the role of rotation in the disc's stability.

(f) Finally, let $\nu \neq 0$ and $N^2 \neq 0$. Consider small-scale modes so that $\Omega/(k^2\nu) \equiv \epsilon \ll 1$.

Expand the growth rate so that $s = \epsilon\Omega s_1 + \dots$ and determine s_1. Find the instability criterion for this mode and compare with part (e). Discuss the role of viscosity in the disc's stability.

END OF PAPER