UNIVERSITY OF CAMBRIDGE

MAMA/318, NST3AS/318, MAAS/318

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 9:00 am to 11:00 am

PAPER 318

APPROXIMATION THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $E_n(f)$ be the value of the best approximation of a 2π-periodic function $f \in C(\mathbb{T})$ by trigonometric polynomials $t_n \in \mathcal{T}_n$ of degree n with respect to the max-norm $\|g\| := \max_{x\in\mathbb{T}} |g(x)|$, and let $\omega(f, \delta)$ be the modulus of continuity of f.

(a) State the Chebyshev alternation theorem for a trigonometric polynomial $p_n^* \in \mathcal{T}_n$ to be a polynomial of best approximation to $f \in C(\mathbb{T})$.

Prove that, for the Weierstrass function

$$g(x) := \sum_{k=0}^{\infty} c_k \cos 5^k x, \quad \text{where} \quad c_k > 0, \quad \sum_{k=0}^{\infty} c_k < \infty\,,$$

the polynomial $p_n^* \in \mathcal{T}_n$ of best approximation to g is the partial sum

$$p_n^*(x) = \sum_{k=0}^{m} c_k \cos 5^k x\,, \qquad 5^m \leqslant n < 5^{m+1}\,.$$

(b) State the inverse theorem for trigonometric approximation and show that

$$E_n(f) = \mathcal{O}(n^{\alpha}) \quad \text{implies} \quad \omega(f, \tfrac{1}{n}) = \begin{cases} \mathcal{O}(n^{-\alpha}), & 0 < \alpha < 1, \\ \mathcal{O}(\frac{\ln n}{n}), & \alpha = 1\,. \end{cases}$$

Hence find the order of $\omega(g, \frac{1}{n})$ for the Weierstrass function g in (a) when $c_k = 1/a^k$, $1 < a \leqslant 5$.

(c) For the case $c_k = 1/5^k$ in (a), show that for all $n \in \mathbb{N}$ we have

$$|g(x + \tfrac{1}{n}) - g(x)| \geqslant \tfrac{c \ln n}{n}, \qquad \text{for } x = \tfrac{\pi}{2}\,.$$

Hence prove that, in this case,

$$\omega(g, \tfrac{1}{n}) \geqslant \tfrac{c \ln n}{n}.$$

Explain briefly why the class of functions f with $E_n(f) = \mathcal{O}(\frac{1}{n})$ cannot be characterized in terms of $\omega(f, \frac{1}{n})$.

2

For a 2π-periodic function $f \in C(\mathbb{T})$, let $s_n(f)$ be its partial Fourier sum of degree n, and let $\sigma_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} s_k(f)$ be its Fejer sum of degree $n-1$. Further, let $\|f\|$ be the max-norm of f on $\mathbb{T} := [-\pi, \pi]$.

(a) From the definition of s_n derive the integral representation

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t) f(t)\, dt, \qquad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}\,.$$

(b) Using the following expression for the Fejer kernel F_n

$$\sigma_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} F_n(x-t) f(t)\, dt\,, \qquad F_n(x) = \frac{1}{2n}\frac{\sin^2\frac{n}{2}x}{\sin^2\frac{1}{2}x}\,,$$

show that $\frac{1}{\pi}\int_{\mathbb{T}} |F_n(t)|\, dt = 1$, and prove the estimate

$$\|\sigma_n(f) - f\| \leqslant c\,\omega_2(f, \tfrac{1}{\sqrt{n}})\,.$$

Here

$$\omega_2(f,t) := \sup_{0<h\leqslant t}\sup_{x\in\mathbb{T}} |f(x-h) - 2f(x) + f(x+h)|$$

is the second modulus of smoothness of f.

[*Remark.* You may use the property $\omega_2(f,\lambda t) \leqslant (\lambda+1)^2\omega_2(f,t)$ without the proof.]

(c) Prove that if f'' is continuous, then $\omega_2(f,t) \leqslant t^2\|f''\|$, and prove that for such f we have

$$\|\sigma_n(f) - f\| = \mathcal{O}(\tfrac{1}{n})\,.$$

By considering an appropriate $f_0 \in C^2(\mathbb{T})$ show that we cannot have a little-o estimate

$$\|\sigma_n(f) - f\| = o(\tfrac{1}{n})$$

valid for all $f \in C^2(\mathbb{T})$.

3

Given $n, k \in \mathbb{N}$, and a knot sequence $(t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}, \qquad N_i(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1- and L_∞-normalized B-splines, respectively.

(a) Prove that the M_i are piecewise-polynomial functions of degree $k-1$ and global smoothness C^{k-2}, with knots $(t_i, \ldots, t_{i+k})$ and with the finite support $[t_i, t_{i+k}]$.

(b) Using the Leibnitz rule for divided differences (given below in (c)) derive the recurrence formula for B-splines:

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)\,,$$

where $N_{i,m}$ is the L_∞-normalized B-spline of order m with support $[t_i, t_{i+m}]$.

(c) Prove the Leibnitz rule for divided differences: if $h = fg$, then

$$h[t_0 ... t_k] = \sum_{m=0}^{k} f[t_0 ... t_m]\, g[t_m ... t_k]\,.$$

4

(a) State the Korovkin theorem on approximation of functions $f \in C[0,1]$ by positive linear operators.

(b) Let $(N_i)_{i=1}^n$ be a sequence of B-splines of degree $k-1$ on a knot-sequence $(t_i)_{i=1}^{n+k}$, and let $\omega_i(x) = (x - t_{i+1}) \cdots (x - t_{i+k-1})$. From the Marsden identity

$$(x-t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k \leqslant t \leqslant t_{n+1}, \quad \forall x \in \mathbb{R},$$

find the coefficients $(a_{m,i})$ of the B-spline expansion of the monomials

$$t^m = \sum_{i=1}^{n} a_{m,i} N_i(t), \quad t_k \leqslant t \leqslant t_{n+1}, \quad m = 0, \ldots, k-1.$$

(c) For $k \geqslant 3$, let $\mathcal{S}_k(\Delta_n)$ be a sequence of spline spaces of degree $k-1$ on the interval $[0,1]$ with the knot-sequences

$$\Delta_n = \{t_1^{(n)} = \ldots = t_k^{(n)} = 0 < t_{k+1}^{(n)} < \ldots < t_n^{(n)} < t_{n+1}^{(n)} = \ldots = t_{n+k}^{(n)} = 1\}$$

such that $|\Delta_n| := \max_i |t_{i+1}^{(n)} - t_i^{(n)}| \to 0$ as $n \to \infty$. Consider the Schoenberg-type operator

$$V_n : C[0,1] \to \mathcal{S}_k(\Delta_n), \quad V_n(f,t) = \sum_{i=1}^{n} f(\tau_i^{(n)}) N_{i,n}(t),$$

where $(N_{i,n})$ is the B-spline basis for $\mathcal{S}_k(\Delta_n)$ and $\tau_i^{(n)}$ are any points satisfying

$$t_i^{(n)} < \tau_i^{(n)} < t_{i+k}^{(n)}.$$

Using (a) and (b), or otherwise, prove that, for any $f \in C[0,1]$, we have

$$\|V_n(f) - f\|_{C[0,1]} \to 0 \quad (n \to \infty).$$

[*Remark.* In your proof, you may suppress index n in $\tau_i^{(n)}$, $t_i^{(n)}$ and $N_{i,n}$ when there is no ambiguity.]

5

(a) Define an orthonormal wavelet ψ. Define a multiresolution analysis of $L_2(\mathbb{R})$ with a generator ϕ, and state the Meyer-Mallat theorem about existence of an orthonormal wavelet ψ.

(b) Prove that the following two properties of ϕ

$$1)\ \ \phi(x) = \sum_n a_n \phi(2x - n), \qquad 2)\ \ \{\phi(\cdot - n)\}_{n \in \mathbb{Z}} \text{ is an orthonormal sequence}$$

are equivalent, respectively, to

$$1')\ \ f(2t) = m(t) f(t)\,, \qquad m(t) = \frac{1}{2} \sum_n a_n e^{-int},$$

$$2')\ \ \sum_k |f(t + 2\pi k)|^2 \equiv 1 \text{ a.e.},$$

where f is the Fourier transform of ϕ, i.e., $f(t) = \widehat{\phi}(t) = \int_{\mathbb{R}} \phi(x) e^{-ixt}\, dx$.

(c) Verify that conditions 1′) - 2′) are fulfilled for the function f defined as

$$f(t) = \begin{cases} 1, & t \in [-\pi, \pi)\,, \\ 0, & \text{otherwise.} \end{cases}$$

Hence determine the corresponding generator ϕ and the coefficients a_n in the equality

$$\phi(x) = \sum_n a_n \phi(2x - n)\,.$$

END OF PAPER