MAMA/317, NST3AS/317, MAAS/317, PCAYM3/317

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 9:00 am to 12:00 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a cluster of chemically homogeneous massive stars composed of hydrogen and helium with negligible metals. They remain fully mixed but radiative as they evolve. The stellar material behaves as an ideal gas and radiation pressure is negligible. Hydrogen burning generates energy at a rate per unit mass $\epsilon = \epsilon_0 X \rho T^{13}$. The opacity, $\kappa = \kappa_0(1+X)$ depends only on the hydrogen mass fraction X. Show that these stars are homologous with the zero-age main-sequence radii

$$R \propto M^{3/4}$$

and luminosities

$$L \propto M^3,$$

where M is the mass of the star.

Determine the slope of this zero-age main sequence and sketch it in an Hertzsprung–Russell diagram.

The stars remain fully mixed for their entire hydrogen-burning lifetimes eventually becoming pure helium stars. During this time they lose negligible mass. Show that, for a given mass M, their luminosity varies with X such that

$$L \propto (1+X)^{-1}(3+5X)^{-4}$$

and determine the similar relation for R.

Suppose the stars are initially composed of pure hydrogen. Find the logarithmic gradient of luminosity with effective temperature $T_{\rm e}$

$$\frac{\mathrm{d}\log L}{\mathrm{d}\log T_{\rm e}}$$

as X starts to fall at the zero-age main sequence and indicate on the same Hertzsprung–Russell diagram the direction in which stars start to evolve away from the main sequence.

2

From the Euler momentum equation in the form

$$\ddot{\boldsymbol{r}} = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot\mathbb{P} + \boldsymbol{F},$$

where $\boldsymbol{r}$ is the position vector of a fluid element of density ρ, $\mathbb{P}$ is the symmetric stress tensor and $\boldsymbol{F}$ is the body force acting on the element, derive the virial theorem for a star in the form

$$\frac{1}{2}\ddot{I} = 2T + 3\int_V P\,\mathrm{d}V - 3P_{\mathrm{s}}V + \mathit{\Omega},$$

where P is pressure equal to P_{S} at the surface and V is the volume of the star. Explain the meanings of the terms I, T and $\mathit{\Omega}$.

A star of total mass M and radius R has a small isothermal core of mass M_{c} and radius $R_{\mathrm{c}} \ll R$ enclosed within a deep convective envelope. The equation of state is that of an ideal gas throughout both the core and envelope. Nuclear burning is taking place just outside the core. The pressure at the surface of the core is P_{c} and its temperature is T_{c}. Use the virial theorem in hydrostatic equilibrium to show that

$$P_{\mathrm{c}} = \lambda\frac{M_{\mathrm{c}}T_{\mathrm{c}}}{R_{\mathrm{c}}^3} - \eta\frac{M_{\mathrm{c}}^2}{R_{\mathrm{c}}^4},$$

where $\lambda > 0$ and $\eta > 0$ are structural constants.

Explain briefly why the nature of nuclear reactions mean that T_{c} is approximately constant.

Deduce that there is a maximum surface pressure $P_{\mathrm{c,max}}$ corresponding to a core radius $R_{\mathrm{c,max}}$.

Now assume that the envelope behaves homologously and show that

$$P_{\mathrm{c}} \propto \frac{T_{\mathrm{c}}^4}{M^2}.$$

Deduce that this core and envelope structure can exist only if

$$M_{\mathrm{c}} < M_{\mathrm{crit}},$$

where the critical core mass

$$M_{\mathrm{crit}} \propto M,$$

the total mass of the the star.

3 A spherical star is composed of a gas with equation of state

$$P = K\rho^2,$$

relating its pressure P to its density ρ for a constant K. Show that the radius of the star is

$$R = \left(\frac{K\pi}{2G}\right)^{\frac{1}{2}}$$

and that the ratio of its mean density $\bar{\rho}$ to its central density ρ_{c} is

$$\frac{\bar{\rho}}{\rho_{\mathrm{c}}} = \frac{3}{\pi^2}.$$

A cubic star of volume L^3 is composed of the same material. Show that it is possible to construct a solution to the structure equations for $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant L$ and $0 \leqslant z \leqslant L$ such that ρ vanishes on the faces of the cube. Show that, for this cubic star,

$$\frac{\bar{\rho}}{\rho_{\mathrm{c}}} = \frac{8}{\pi^3}.$$

Comment briefly on whether you expect such a star to exist in nature both theoretically and observationally.

4

Consider a semi-detached binary star system, with conservative Roche lobe overflow. Show that the separation a of the circular orbit satisfies

$$a \propto \frac{(1+q)^4}{q^2},$$

where $q = M_1/M_2$ is the mass ratio of the two components, the donor star 1 and accretor star 2.

In a range of mass ratios of interest the radius of the Roche lobe around the donor can be approximated by

$$R_{\mathrm{L}} \approx 0.4 a q^{2/9}.$$

Show that, as the donor conservatively transfers mass to its companion, its Roche-lobe radius changes at a rate

$$\frac{\mathrm{d}\log R_{\mathrm{L}}}{\mathrm{d}\log M_1} = \alpha,$$

where

$$\alpha = \frac{20}{9}\left(q - \frac{4}{5}\right).$$

On the main sequence the radius R of a star of mass M and age t varies according to

$$\log_e R = \beta \log_e M + \frac{t}{\tau_{\mathrm{nuc}}},$$

on a long nuclear timescale τ_{nuc}. Both β and τ_{nuc} are constant. While $R_1 < R_{\mathrm{L}}$ the mass M_1 remains constant but when $R_1 > R_{\mathrm{L}}$ mass transfer takes place at a rate given by

$$-\frac{\mathrm{d}\log M_1}{\mathrm{d}t} = \frac{1}{\tau_{\mathrm{dyn}}}\log\left(\frac{R_1}{R_{\mathrm{L}}}\right),$$

where the constant dynamical timescale $\tau_{\mathrm{dyn}} \ll \tau_{\mathrm{nuc}}$. Let $f = \log_e(R_1/R_{\mathrm{L}})$. While $f < 0$ show that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{\tau_{\mathrm{nuc}}}$$

and find a corresponding first-order linear differential equation satisfied by f when $f > 0$.

Show that, as long as $\beta > \alpha$,

$$f \to \frac{1}{\beta - \alpha}\frac{\tau_{\mathrm{dyn}}}{\tau_{\mathrm{nuc}}}.$$

Explain why this corresponds to steady mass transfer on a nuclear timescale.

How does f evolve if $\beta < \alpha$?

END OF PAPER