UNIVERSITY OF CAMBRIDGE

MAMA/316, NST3AS/316, MAAS/316, PCAYM3/316

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2025 9:00 am to 12:00 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Consider a belt of spherical planetesimals with diameters between D_{min} to $D_{\mathrm{max}} \gg D_{\mathrm{min}}$. The number of planetesimals with diameters D to $D + dD$ is $n(D)dD = KD^{-\alpha}dD$, where α is a constant in the range 3 to 4. Determine the normalisation constant K in terms of the total mass in the belt M and the planetesimal density ρ.

(ii) It can be assumed that the volume density of planetesimals is uniform within the volume V that the belt occupies, that planetesimals encounter each other at relative velocities v_{rel}, and that gravitational focussing can be ignored. Show that the rate of collisions between a planetesimal of size D and those in the size range D_{im} to $D_{\mathrm{im}} + dD_{\mathrm{im}}$ is given by

$$R_{\mathrm{col}}(D, D_{\mathrm{im}})dD_{\mathrm{im}} = AD_{\mathrm{im}}^{2-\alpha}(1 + D/D_{\mathrm{im}})^2 dD_{\mathrm{im}},$$

where the constant A should be determined.

(iii) The specific incident energy Q relative to the dispersal threshold $Q_{\mathrm{D}}^{\star}$ determines the outcome of a collision of an impactor of size D_{im} onto a target of size D. Give an expression for the size of impactor $X_{\mathrm{c}}D$ above which collisions become catastrophic and derive the rate of catastrophic collisions for planetesimals in the belt, showing that for $X_{\mathrm{c}} \ll 1$ this is approximately

$$R_{\mathrm{cc}}(D) \approx A(\alpha - 1)^{-1} X_{\mathrm{c}}^{1-\alpha} D^{3-\alpha}.$$

(iv) A cratering collision is one with $Q < Q_{\mathrm{D}}^{\star}$, for which the largest remnant after the collision has a mass $f_{\mathrm{lr}} = 1 - 0.5Q/Q_{\mathrm{D}}^{\star}$ times that of the target m. Show that the fraction of target mass lost in a cratering collision is $0.5[D_{\mathrm{im}}/(X_{\mathrm{c}}D)]^3$.

(v) Derive an expression for the rate of mass loss from the target due to cratering collisions, $\dot{m}_{\mathrm{cr}}$, and so show that the charateristic timescale for mass loss due to cratering collisions is approximately

$$m/\dot{m}_{\mathrm{cr}} \approx [2(4 - \alpha)/(\alpha - 1)]/R_{\mathrm{cc}}(D).$$

(vi) For catastrophic collisions the largest remnant after the collision has a mass $f_{\mathrm{lr}} = 0.5(Q_{\mathrm{D}}^{\star}/Q)^{1.24}$. Similar to (v), derive the characteristic timescale for mass loss due to catastrophic collisions.

(vii) For a steady state collisional cascade in which the dispersal threshold is independent of size, is mass loss dominated by catastrophic or cratering collisions?

2

(i) The equation of motion for a dust particle acted on by radiation pressure and stellar gravity is $\ddot{\boldsymbol{r}} + \mu(1-\beta)\boldsymbol{r}/r^3 = 0$, where $\boldsymbol{r}$ is the vector from the star to the particle, r is the magnitude of that vector, $\mu = GM_\star$, $M_\star$ is the star's mass, and β is the particle's radiation pressure coefficient. The dust particle in question has $\beta \gg 1$. Derive the two constants of motion, the vector $\boldsymbol{h}$ and C, that are associated with the angular momentum and energy of the particle's orbit, respectively, and show that this orbit is confined to a plane.

(ii) Considering motion in the plane show that the path of the orbit can be derived to be

$$r = [h^2/(\mu(1-\beta))]/[1 + e\cos(\theta - \varpi)],$$

where $h = |\boldsymbol{h}|$, θ is the azimuthal angle, and e and ϖ are constants of integration.

(iii) The particle is released with zero relative velocity from a parent body that was large enough to be unaffected by radiation pressure, and which was on a circular orbit around the star at a distance $a_{\rm p}$. Determine the constants h and e for the particle's orbit and describe what the angle ϖ represents.

(iv) Derive the particle's velocity a long time after it was released (i.e., after many orbital periods of the parent body), and show that at this point it would be found at an azimuthal angle $\sim \sqrt{2/\beta}$ radians from the point at which it was released.

(v) Derive an expression for the radial velocity of the particle as a function of its distance from the star.

(vi) The parent body is continuously releasing dust grains with the same β. Use the result from (v) to sketch the surface density of the resulting disk of dust particles as a function of distance.

(vii) The dust surface density is observed at large distances from the star and extrapolated inwards assuming a power law dependence on distance. Derive how much higher the actual surface density is at a radial distance of $2a_{\rm p}$ than would be predicted by this extrapolation.

3

(i) Consider a coplanar planetary system comprised of 2 planets on well separated orbits, i.e., $a_2 \gg a_1$, where a_j is the semimajor axis of the j-th planet. With reference to the disturbing function, describe the different type of perturbations that may affect the evolution of the planets' orbits, as well as the physical interpretation of those perturbations.

(ii) The planets' orbits have low eccentricities $e_j \ll 1$ and are not near a mean motion resonance. Combining their complex eccentricities $z_j = e_j \exp(i\varpi_j)$ into a vector $\boldsymbol{z} = [z_1, z_2]$, it can be shown that $\dot{\boldsymbol{z}} = i\boldsymbol{A}\boldsymbol{z}$, where ϖ_j is the longitude of pericentre of planet j, and the 2 by 2 matrix $\boldsymbol{A}$ has elements that depend on the masses and semimajor axes of the planets' orbits. Derive the solution to this evolution

$$z_j = \sum_{k=1}^{2} e_{jk} \exp\left[i(g_k t + \beta_k)\right],$$

describing in detail how to determine the constants e_{jk}, g_k and β_k.

(iii) Consider a test particle on a coplanar orbit within this planetary system. The evolution of the particle's complex eccentricity z due to perturbations from the planets is given by

$$\dot{z} = iAz + i\sum_{j=1}^{2} A_j z_j,$$

where A, A_1 and A_2 are constants that depend on the masses and semimajor axes of the planets as well as the particle's semimajor axis a. Derive the evolution of the particle's complex eccentricity and describe its behaviour.

(iv) Sketch how you expect A to depend on the particle's semimajor axis, and hence why there must be either 2, 3 or 4 locations in the planetary system where $A = g_1$.

(v) The particle's orbit is also subject to a dissipative force which damps its eccentricity e on a timescale τ at a rate proportional to eccentricity, i.e., there is an additional perturbation which can be written $\dot{e} = -e/\tau$. Derive the solution to the evolution in this case and describe qualitatively how that differs to the case without dissipation.

(vi) Consider the particle's orbit on timescales $t \gg \tau$. What does this orbit tend to for particles at locations for which $A = g_1$ and how is that different to particles that are not subject to the dissipation?

(vii) Explain without detailed calculation what you expect the particle's orbit to be for $t \gg \tau$ close to one of the planets.

4

(i) A planetesimal is orbiting a star of mass $M_\star$. The orbit is just bound, with eccentricity $e \approx 1$. Its pericentre distance q is very close to that of a planet of mass $M_{\mathrm{p}} \ll M_\star$ which is on a coplanar circular orbit at a distance a_{p}. Determine the distance from the star at which the planetesimal is moving at the same angular speed as the planet, the approximate true anomaly of the planetesimal at this point (to the nearest 10 deg), and its approximate angular separation from the planet (as seen from the star).

(ii) The two objects have a conjunction very close to the planetesimal's pericentre. Ignoring any perturbation to the orbit due to the encounter, sketch the orbit of the planetesimal in the frame rotating with the planet.

(iii) Consider the perturbation to the planetesimal's orbit as a hyperbolic encounter in which stellar gravity can be ignored, with this interaction starting at the planetesimal's pericentre when both objects are moving azimuthally. In the planetocentric frame, this interaction deflects the planetesimal's velocity by an angle θ, which you can assume to be given by

$$\sin(\theta/2) = [1 + b^2 v_\infty^4/(GM_{\mathrm{p}})^2]^{-1/2},$$

for an impact parameter b and incoming velocity in this frame of v_∞. If the pericentre is just outside the planet's orbit at $q = a_{\mathrm{p}}(1+\delta)$ near its L2 Lagrange equilibrium point, show that $\theta \approx A(\mu/\delta)$, where $\mu = M_{\mathrm{p}}/M_\star$ and A should be determined.

(iv) Sketch the path of the encounter in the planetocentric frame and the change in the planetesimal's velocity vector in the inertial frame. Hence show that the square of its velocity in the inertial frame changes by

$$\Delta(v^2) = -B(\mu/\delta)^2,$$

where B should be determined.

(v) Comment on how the semimajor axis of the planetesimal's orbit has changed as a result of the encounter and derive an expression for how its pericentre distance has changed.

(vi) Discuss the expected evolution of the planetesimal's orbit on long timescales.

END OF PAPER