MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 1:30 pm to 4:30 pm

PAPER 315

EXTRASOLAR PLANETS: ATMOSPHERES AND INTERIORS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. Consider a plane parallel atmosphere in local thermodynamic equilibrium (LTE), with a pressure-temperature profile $T = T(P)$ and a source of radiation at the bottom of the atmosphere. Consider the specific intensity $\mathrm{I}_{\nu,1}$ at a pressure P_1 deep in the atmosphere.

Starting with the radiative transfer equation derive an expression for the specific intensity at a pressure P_2 in the atmosphere, where $P_2 < P_1$. Express your answer in terms of P_1, P_2, the Planck function and any other quantities required. Make and state any assumptions needed. Using this result derive the corresponding expression for an isothermal atmosphere.

b. Derive the expression for the eclipse depth of a transiting exoplanet observed during secondary eclipse. Assume the planet and star to be emitting as blackbodies with temperatures T_p and T_s, and to have radii R_p and R_s, respectively.

Based on the above result, sketch a qualitative plot of the planet-star flux ratio as a function of wavelength for a typical hot Jupiter. Explain what happens in the long wavelength (Rayleigh-Jeans) limit.

c. The emergent spectrum of an exoplanet was found to have two peaks at wavelengths of 0.5 μm and 1 μm. What could be causing the two peaks? The planet-star flux ratio of the spectrum in the far infrared was found to be 0.1% and the star was found to have a radius that is half that of the Sun with a similar age. What is the approximate radius of the planet relative to that of Jupiter? Make and state any assumptions required. Which observational method is most likely to characterise the atmosphere of such a planet and why?

2

a. A sub-Neptune exoplanet Cam-1 b orbits an M dwarf star with a radius half that of the Sun. The planet has a mass of 5.0 ± 0.2 Earth masses and a radius of 2.0 ± 0.1 Earth radii, with a density consistent with that expected for a 100% H_2O interior. Discuss three scenarios for the possible internal structure of the planet. Discuss how atmospheric observations may be used to break the degeneracies between the three scenarios.

b. Assume that Cam-1 b is found to host a H_2-rich atmosphere with an equilibrium temperature of 300 K and orbits in the habitable zone of its star. What are the atmospheric conditions required for the planet to be potentially habitable? Estimate the expected amplitude of a molecular spectral feature that could be observed in the transmission spectrum of the planet with JWST. Make and state any assumptions needed.

c. Derive the Lane-Emden equation for a polytropic model of planetary structure which is of the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n, \tag{1}$$

where n is the polytropic index. Show that the total mass and radius for such a model can be related as

$$M^\alpha R^\beta = C, \tag{2}$$

where M and R are the total mass and radius of the planet and C is a constant. Express α and β as a function of n. What physical situations in planetary interiors can be described by polytropic indices $n = 0$ and n=3/2? Give an example of an astronomical object to which each index applies.

d. Consider a rocky planet with a mass M_p, surface radius R_p, and a density ρ_c that is constant throughout the planet's interior. Derive an expression for the pressure P in the interior as a function of radial distance r from the centre of the planet, the surface gravity g, surface radius, and density.

3

a. Consider a hypothetical habitable zone (HZ) for an Earth-like planet around main-sequence stars. Assume that an equilibrium temperature between T_{min} and T_{max} is required in the atmosphere to sustain liquid water on the planetary surface. Further assume that the stellar luminosity scales with the stellar mass as $L_{\mathrm{S}} = K M_{\mathrm{S}}^{\beta}$. Construct the HZ for such a planet by determining the habitable orbital separation (a_{H}) as a function of the stellar mass. Make and state any assumptions needed.

b. What are the locations of the boundaries of the HZ around the sun for an Earth-like planet and what are the processes that govern them? Make and state any assumptions needed.

c. What are three current practical challenges in observing atmospheres of Earth-like exoplanets?

d. Discuss the essential requirements for a planet to be habitable, providing examples for each. Discuss four other factors that can affect planetary habitability.

e. Discuss three types of habitable exoplanets that may exist in nature considering the following factors for each of them: (a) internal structure, (b) atmospheric composition, (c) potential biosignatures, (d) observability with current and/or upcoming observational facilities.

4

a. Using the radiative transfer equation show how the temperature gradient in an atmosphere can affect its spectral features. Discuss the possible causes of thermal inversions in the following three types of planets: (a) hot jupiters, (b) solar system planets, and (c) sub-Neptunes orbiting M dwarfs. Explain how a thermal inversion can affect the habitability of a temperate planet and provide an example.

b. A cold giant planet orbiting a sun-like star with an equilibrium temperature of 100 K has a thermal inversion in the stratosphere at pressures below 0.1 bar. Derive an expression for the possible temperature gradient in the lower atmosphere below the stratosphere. Make and state any assumptions needed.

c. Discuss three aspects of atmospheric dynamics that have been observed or predicted for giant exoplanets. Compare them with those of solar system giant planets.

d. Discuss three signatures of non-equilibrium chemistry observable in exoplanetary atmospheres. Discuss how the atmospheric compositions of solar system planets help construct a planetary mass-metallicity relation and provide constraints on planetary formation. How would atmospheric observations of exoplanets help in this direction?

e. Discuss three current or upcoming observational facilities for characterising exoplanetary atmospheres, noting the observing method, instrumental capability, and key results if any. What makes JWST better than previous observational facilities? State three advancements or future directions in exoplanetary science that have been made possible by JWST.

END OF PAPER