UNIVERSITY OF CAMBRIDGE

MAMA/314, NST3AS/314, MAAS/314

MAT3

**MATHEMATICAL TRIPOS Part III**

Friday 6 June 2025 1:30 pm to 4:30 pm

# PAPER 314

# ASTROPHYSICAL FLUID DYNAMICS

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial\rho}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\rho=-\rho\boldsymbol{\nabla}\cdot\mathbf{u}, \tag{1}$$

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}p=-\gamma p\boldsymbol{\nabla}\cdot\mathbf{u}, \tag{2}$$

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)=-\rho\boldsymbol{\nabla}\Phi-\boldsymbol{\nabla}p+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}, \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}), \tag{4}$$

$$\nabla^2\Phi=4\pi G\rho. \tag{5}$$

You may assume the following results from vector calculus. For vector fields $\mathbf{F}$ and $\mathbf{G}$:

$$(\boldsymbol{\nabla}\times\mathbf{F})\times\mathbf{F}=\mathbf{F}\cdot\boldsymbol{\nabla}\mathbf{F}-\boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{F}|^2\right), \tag{6}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot(\boldsymbol{\nabla}\times\mathbf{F})-\mathbf{F}\cdot(\boldsymbol{\nabla}\times\mathbf{G}), \tag{7}$$

$$\boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\boldsymbol{\nabla}\mathbf{F}-\mathbf{F}\cdot\boldsymbol{\nabla}\mathbf{G}-\mathbf{G}(\boldsymbol{\nabla}\cdot\mathbf{F})+\mathbf{F}(\boldsymbol{\nabla}\cdot\mathbf{G}). \tag{8}$$

For scalar and vector fields $f$ and $\mathbf{F}$, in cylindrical polar coordinates $(r,\phi,z)$:

$$\boldsymbol{\nabla}f=\frac{\partial f}{\partial r}\,\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial\phi}\,\mathbf{e}_\phi+\frac{\partial f}{\partial z}\,\mathbf{e}_z, \tag{9}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r}\frac{\partial(rF_r)}{\partial r}+\frac{1}{r}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}, \tag{10}$$

$$\boldsymbol{\nabla}\times\mathbf{F}=\left(\frac{1}{r}\frac{\partial F_z}{\partial\phi}-\frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\mathbf{e}_\phi+\frac{1}{r}\left(\frac{\partial(rF_\phi)}{\partial r}-\frac{\partial F_r}{\partial\phi}\right)\mathbf{e}_z. \tag{11}$$

In spherical polar coordinates $(r,\theta,\phi)$:

$$\boldsymbol{\nabla}f=\frac{\partial f}{\partial r}\,\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\,\mathbf{e}_\phi, \tag{12}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}, \tag{13}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F}=&\frac{1}{r\sin\theta}\left(\frac{\partial(F_\phi\sin\theta)}{\partial\theta}-\frac{\partial F_\theta}{\partial\phi}\right)\mathbf{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{1}{r}\frac{\partial(rF_\phi)}{\partial r}\right)\mathbf{e}_\theta\\&+\frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\phi.\end{aligned} \tag{14}$$

**1**

(a) When can an astrophysical fluid flow reasonably be treated as the adiabatic flow of a perfect gas with adiabatic exponent $\gamma = 5/3$?

(b) Consider the steady, adiabatic, spherically symmetric accretion of a perfect gas with $\gamma = 5/3$ on to a body of mass $M$. The body can be regarded as a Newtonian point mass that absorbs any incoming gas (similarly to a black hole). At large distance from the body, the gas can be considered to be essentially at rest and to have uniform density $\rho_0$ and adiabatic sound speed $v_{s0}$. The self-gravity of the gas can be neglected. Show that $\rho \propto v_s^3$ and that the Mach number $\mathcal{M}(r)$ of the accretion flow is related to the radial coordinate $r$ by

$$A\dot{M}^{1/2}\left(\mathcal{M}^{3/2} + 3\mathcal{M}^{-1/2}\right) = GM + Br,$$

where $\dot{M}$ is the accretion rate and $A$ and $B$ are positive constants that you should express in terms of $\rho_0$ and $v_{s0}$.

(c) Show that the flow does not become supersonic.

(d) Explain why the solution is valid for all $r > 0$ only if the accretion rate does not exceed the critical value

$$\dot{M}_{\text{crit}} = \frac{\pi\rho_0(GM)^2}{v_{s0}^3}.$$

(e) Define $\dot{m} = \dot{M}/\dot{M}_{\text{crit}}$ and consider the approximate behaviour of the flow in the limit $r \ll GM/B$, assuming that $\dot{m} \leqslant 1$. Show that the Mach number approaches a constant value $\mathcal{M}_1$ and find the algebraic relation between $\dot{m}$ and $\mathcal{M}_1$. Determine the scaling of the density and sound speed with $r$ in this limit and show that the fluid velocity is approximately

$$\left(\frac{\dot{m}\mathcal{M}_1^3}{4}\right)^{1/4}\left(\frac{GM}{r}\right)^{1/2}.$$

**2**

A model 'star' consists of a perfect gas in hydrostatic equilibrium in the gravitational potential

$$\Phi = \frac{1}{2}\Omega^2 r^2,$$

where $\Omega$ is a positive constant and $(r, \theta, \phi)$ are spherical polar coordinates. In this question the potential $\Phi$ is regarded as fixed and the self-gravity of the star is neglected.

(a) Write down the relation between pressure and density that holds in hydrostatic equilibrium. Solve it in the case of a polytropic model with

$$p = K\rho^{1+1/n},$$

where $K$ and $n$ are positive constants, and express the density and pressure as functions of $r$ for a model star (in the fixed gravitational potential) with outer radius $R$.

(b) Formulate the linearized equations for small perturbations of the star, again neglecting its self-gravity. Consider oscillation modes in which the radial displacement is of the form

$$\xi_r = \mathrm{Re}\left[\tilde{\xi}_r(r)\, Y_\ell^m(\theta,\phi)\, e^{-i\omega t}\right],$$

and similarly for the Eulerian perturbations $\delta\rho$ and $\delta p$, while the horizontal displacement is of the form

$$\boldsymbol{\xi}_\mathrm{h} = \mathrm{Re}\left[\tilde{\xi}_\mathrm{h}(r)\, \boldsymbol{\nabla} Y_\ell^m(\theta,\phi)\, e^{-i\omega t}\right],$$

where $Y_\ell^m(\theta,\phi)$ is one of the spherical harmonic functions (with integers $\ell$ and $m$ such that $\ell \geqslant |m|$) satisfying

$$\nabla^2 Y_\ell^m = -\frac{\ell(\ell+1)}{r^2} Y_\ell^m.$$

Show that the wave amplitudes $\tilde{\xi}_r$, $\tilde{\xi}_\mathrm{h}$, $\tilde{\delta\rho}$ and $\tilde{\delta p}$ (on which the tildes are henceforth omitted) satisfy

$$\begin{aligned}
-\rho\omega^2\xi_r &= -g\,\delta\rho - \frac{d\,\delta p}{dr},\\
-\rho\omega^2\xi_\mathrm{h} &= -\delta p,\\
\delta\rho &= -\xi_r\frac{d\rho}{dr} - \rho\Delta,\\
\delta p &= -\xi_r\frac{dp}{dr} - \gamma p\Delta,
\end{aligned}$$

where $g = \Omega^2 r$ is the inward radial gravity, $\gamma$ is the adiabatic exponent and

$$\Delta = \frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} - \frac{\ell(\ell+1)}{r^2}\xi_\mathrm{h}.$$

(c) Verify that there is an oscillation mode with $\Delta = 0$ and $\xi_r \propto r^{\ell-1}$, for any integer $\ell \geqslant 1$. Obtain an expression for the squared frequency $\omega_\ell^2$ of this mode and describe its physical nature.

**[QUESTION CONTINUES ON THE NEXT PAGE]**

(d) Now consider a problem in which the star is tidally forced by a massive companion in a circular orbit and therefore experiences (in addition to the fixed potential $\Phi$) a tidal potential

$$\Psi = \mathrm{Re}\left[A r^{\ell}\, Y_{\ell}^{m}(\theta,\phi)\, e^{-i\omega t}\right],$$

where $A$ and $\omega$ are real constants representing the amplitude and angular frequency of the tidal forcing. (Note that $\omega$ is now externally imposed and different from the frequency of a free oscillation mode considered in the previous parts.) Show that the periodic response of the star to this forcing involves the oscillation mode described in part (c) such that the radial displacement is

$$\mathrm{Re}\left[\left(\frac{A}{\omega^2-\omega_{\ell}^2}\right)\ell r^{\ell-1}\, Y_{\ell}^{m}(\theta,\phi)\, e^{-i\omega t}\right],$$

where $\omega_\ell$ is the natural frequency of the mode found in part (c). Comment on the dependence of the amplitude of the forced response on the tidal forcing frequency $\omega$.

**3**

Consider a static equilibrium state of uniform density $\rho$, pressure $p$ and magnetic field $\mathbf{B}$, in the absence of gravity. You may assume that small perturbations of this state are governed by the linearized equation of motion

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla}\delta\Pi + \frac{1}{\mu_0}\mathbf{B}\cdot\boldsymbol{\nabla}\delta\mathbf{B},$$

with

$$\delta\Pi = -\gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0} \qquad \text{and} \qquad \delta\mathbf{B} = \mathbf{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}),$$

where $\boldsymbol{\xi}$ is the displacement and $\delta$ denotes an Eulerian perturbation. You may also assume that, for plane waves in which the displacement is of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)\right]$$

(and similarly for all Eulerian perturbations), where $\tilde{\boldsymbol{\xi}}$ and $\mathbf{k}$ are constant vectors and $\omega$ is a constant, the wave amplitude $\tilde{\boldsymbol{\xi}}$ satisfies the algebraic equation

$$\omega^2\tilde{\boldsymbol{\xi}} = \left[(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)(\mathbf{k}\cdot\tilde{\boldsymbol{\xi}}) - (\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}})(\mathbf{v}_{\mathrm{a}}\cdot\tilde{\boldsymbol{\xi}})\right]\mathbf{k} + (\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}})^2\tilde{\boldsymbol{\xi}} - (\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}})(\mathbf{k}\cdot\tilde{\boldsymbol{\xi}})\mathbf{v}_{\mathrm{a}}, \qquad (*)$$

where $v_{\mathrm{s}}$ is the adiabatic sound speed, $\mathbf{v}_{\mathrm{a}}$ is the Alfvén velocity and $v_{\mathrm{a}} = |\mathbf{v}_{\mathrm{a}}|$.

(a) Define the adiabatic sound speed and Alfvén velocity appearing in equation $(*)$.

(b) Deduce from equation $(*)$ that the dispersion relations for MHD waves are

$$\omega^2 = (\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}})^2 \qquad \text{and} \qquad \omega^4 - (v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)|\mathbf{k}|^2\omega^2 + v_{\mathrm{s}}^2|\mathbf{k}|^2(\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}})^2 = 0.$$

Discuss briefly the physical interpretation of the three different MHD wave modes.

(c) Suppose that $\mathbf{B} = B\,\mathbf{e}_x$ and the wave is in the $xz$-plane such that $k_y = 0$. Rearrange the second dispersion relation in part (b) to show that

$$k_z^2 = \frac{(v^2 - v_{\mathrm{s}}^2)(v^2 - v_{\mathrm{a}}^2)}{(v^2 - v_{\mathrm{t}}^2)(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)}k_x^2,$$

where $v = \omega/k_x$ is the phase speed of the wave in the $x$-direction and $v_{\mathrm{t}}$ is a speed (which you should find) smaller than both $v_{\mathrm{s}}$ and $v_{\mathrm{a}}$. Assuming that $\omega$ and $k_x$ are real, determine the ranges of $v^2$ for which $k_z$ is (i) real or (ii) imaginary.

(d) Use the $z$-component of the equation of motion to find an expression for the ratio $\widetilde{\delta\Pi}/\tilde{\xi}_z$ in terms of $\rho$, $v_{\mathrm{a}}$, $v$, $k_x$ and $k_z$, for a wave of the type considered in part (c).

**[QUESTION CONTINUES ON THE NEXT PAGE]**

(e) Consider a static equilibrium state consisting of two uniform regions of the type considered in part (c), separated by an interface at $z = 0$:

$$\mathbf{B} = \begin{cases} B_+ \, \mathbf{e}_x, & z > 0, \\ B_- \, \mathbf{e}_x, & z < 0, \end{cases} \qquad \rho = \begin{cases} \rho_+, & z > 0, \\ \rho_-, & z < 0, \end{cases} \qquad p = \begin{cases} p_+, & z > 0, \\ p_-, & z < 0, \end{cases}$$

where $B_\pm$, $\rho_\pm$ and $p_\pm$ are positive constants such that the total pressure is continuous.

An *interfacial wave* is a solution of the linearized equations on this equilibrium state that decays exponentially with distance from the interface. In each region the wave is therefore of the type considered in part (c), with an imaginary value of $k_z$ of the appropriate sign. The waves in the two regions are to be matched at the interface using appropriate physical conditions.

(i) Explain why the total pressure perturbation and the vertical displacement must be continuous at the interface.

(ii) Use these boundary conditions and the result of part (d) to match the wave solutions in the two regions at the interface.

(iii) Show that the phase speed $v$ of the interfacial wave in the $x$-direction satisfies

$$v^2 = \frac{\Sigma_+ v_{\mathrm{a}+}^2 + \Sigma_- v_{\mathrm{a}-}^2}{\Sigma_+ + \Sigma_-},$$

where $\Sigma_\pm = \rho_\pm / |k_{z\pm}|$, and $k_{z\pm}$ are the imaginary values of $k_z$ in the two regions. Deduce that $v$ lies between the Alfvén speeds in the two regions.

[You may assume that $v$ is in an appropriate range such that $k_z^2 < 0$ in both regions.]

**4**

(a) Explain what is meant by a force-free magnetic field and why it occurs naturally in the limit of very low density in the exterior of an astrophysical body.

(b) Show that the rate of change of the energy of a force-free magnetic field in a volume $V$ bounded by a surface $S$ is given, in ideal MHD, by

$$\frac{1}{\mu_0}\int_S [(\mathbf{u}\times\mathbf{B})\times\mathbf{B}]\cdot d\mathbf{S}.$$

(c) Explain why an axisymmetric magnetic field can be expressed as a sum of poloidal and toroidal parts, $\mathbf{B}=\mathbf{B}_\mathrm{p}+\mathbf{B}_\mathrm{t}$, with the poloidal magnetic field written in terms of a function $\psi(r,z)$ as $\mathbf{B}_\mathrm{p}=\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi$, where $(r,\phi,z)$ are cylindrical polar coordinates, and the toroidal magnetic field being $\mathbf{B}_\mathrm{t}=B_\phi(r,z)\,\mathbf{e}_\phi$. Explain how $\psi$ is related to the distribution of poloidal magnetic flux.

(d) Show that an axisymmetric force-free magnetic field has $rB_\phi=F(\psi)$, where $F(\psi)$ is an arbitrary function of the flux function. Show further that the flux function satisfies the Grad–Shafranov equation

$$-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)-\frac{\partial^2\psi}{\partial z^2}=F\frac{dF}{d\psi}.$$

Determine how $F(\psi)$ is related to the distribution of poloidal electric current.

## END OF PAPER