UNIVERSITY OF CAMBRIDGE

MAMA/313, NST3AS/313, MAAS/313

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 9:00 am to 11:00 am

PAPER 313

SOLITONS, INSTANTONS AND GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi = \phi(x,t)$. Find the Euler–Lagrange equation corresponding to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\Big(\Big(\frac{\partial \phi}{\partial t}\Big)^2 - \Big(\frac{\partial \phi}{\partial x}\Big)^2\Big) - U(\phi).$$

Write down the expression for the total energy corresponding to $\mathcal{L}$, and show that it is conserved from the Euler–Lagrange equations.

From now on set

$$U = \frac{1}{2}(1-\phi^2)^2\phi^4.$$

(i) Let $\phi_{(0,1)}(x)$ be a kink interpolating between two vacua $\phi = 0$ and $\phi = 1$. Find α and β such that $\phi \to \alpha\phi$ and $x \to \beta x$ are discrete symmetries of the field equations, and express all other kinks and anti kinks in terms of $\phi_{(0,1)}$.

(ii) Construct a superpotential W such that

$$U = \frac{1}{2}\Big(\frac{dW}{d\phi}\Big)^2$$

and use the Bogolmolny argument to find the energy of $\phi_{(0,1)}$.

(iii) Show that the kink $\phi \equiv \phi_{(0,1)}$ is implicitly given by

$$\frac{1}{2}\ln\Big(\frac{1+\phi}{1-\phi}\Big) - \frac{1}{\phi} = x - A,$$

where A is a constant.

(iv) Show that

$$\phi_{(0,1)} \sim \begin{cases} \frac{1}{A-x} & \text{as} \quad x \to -\infty \\ 1 - e^{-2(x-b)} & \text{as} \quad x \to \infty \end{cases}$$

where b is a constant which should be determined in terms of A. Sketch the kink profile.

2 Let η be a Euclidean metric on $\mathbb{R}^4$, and let vol be a volume form.

(i) Define the Hodge $\star$–operator of $(\mathbb{R}^4, \eta, vol)$ and show that

$$\star\omega \wedge \star\omega = \omega \wedge \omega \quad \text{where} \quad \omega \in \Lambda^2(\mathbb{R}^4).$$

Deduce that, if $F \in \Lambda^2(\mathbb{R}^4) \otimes \mathfrak{su}(2)$ satisfies $F = -\star F$ then the Yang–Mills action for instantons equals $8\pi^2 c_2$, where c_2 is the second Chern number.

The anti–self–dual Yang–Mills equations on $\mathbb{R}^4$ with the Euclidean metric

$$ds^2 = |dz|^2 + |dw|^2$$

are given by

$$F_{wz} = 0, \quad F_{\bar{w}\bar{z}} = 0, \quad F_{w\bar{w}} + F_{z\bar{z}} = 0 \tag{1}$$

where $F = dA + A \wedge A$, and $A = A_w dw + A_{\bar{w}} d\bar{w} + A_z dz + A_{\bar{z}} d\bar{z}$ take values in a Lie algebra $\mathfrak{g}$ of a Lie group G, and (z, w) are complex coordinates on $\mathbb{R}^4$

(ii) Write down a Lax pair with spectral parameter for (1).

(iii) Show that there exists a gauge such that $A_w = A_z = 0$.

(iv) Use the 2nd equation in (1) to show that there exists a G–valued function $J = J(w, z, \bar{w}, \bar{z})$ such that

$$A = J^{-1}\partial_{\bar{w}} J \, d\bar{w} + J^{-1}\partial_{\bar{z}} J \, d\bar{z},$$

and deduce that the ASDYM system reduces to

$$\partial_w(J^{-1}\partial_{\bar{w}} J) + \partial_z(J^{-1}\partial_{\bar{z}} J) = 0.$$

(v) Hence show that in this gauge the anti–self–dual Maxwell equations on $\mathbb{R}^4$ are equivalent to the Laplace equation for a function which you should specify.

3 Define the topological degree $\deg(\phi)$ of a smooth map ϕ between oriented closed manifolds M_1, M_2, in terms of the volume form ω on M_2.

Consider $\phi : S^2 \to S^2$ with $\phi = (\phi^1, \phi^2, \phi^3) \in \mathbb{R}^3, |\phi| = 1$ and show that

$$\deg(\phi) = \frac{1}{2vol(S^2)} \int \epsilon^{abc} \phi^a d\phi^b \wedge d\phi^c.$$

(i) Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$. Using the method of preimages find $\deg(f)$ if

$$f(z) = \frac{P(z)}{Q(z)} \tag{1}$$

where P and Q are holomorphic polynomials in z of degrees m and n respectively with no common factors (you are not required to show that the method of pre-images agrees with the definition of topological degree).

(ii) Consider the case $Q(z) = 1, P(z) = z^m$ and find $\deg(f)$ using the pull–back of a volume form

$$\frac{idf \wedge d\bar{f}}{(1 + |f|^2)^2}.$$

(iii) Let $\phi : \mathbb{R}^2 \to S^2$ and

$$E[\phi] = \int_{\mathbb{R}^2} \partial_z \phi^a \partial_{\bar{z}} \phi^a idz \wedge d\bar{z} < \infty.$$

Assume that ϕ extends to a compactification ${S^2}_\infty = \mathbb{R}^2 + \{\infty\}$ and show that $E[\phi] \geqslant c|\deg(\phi)|$ where $c > 0$ is a constant which you should find, and the equality is achieved when ϕ satisfies the first order Bogomolny equations which you should derive.

END OF PAPER