UNIVERSITY OF CAMBRIDGE

MAMA/312, NST3AS/312, MAAS/312

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 9:00 am to 12:00 pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In de Sitter spacetime consider a massless scalar field ϕ with interaction

$$S_{int} = \int d^3x d\eta\, a^4\, \frac{\lambda}{3!}\dot{\phi}^3 \,,$$

where $\dot{\phi} = \partial_t \phi$. To compute the four point function of ϕ at $\eta = 0$ induced by this interaction to order λ^2, focus exclusively on the s-channel process $B_{4,s}$ and proceed as follows:

(i) Convert this interaction from cosmological time t to conformal time η.

(ii) State the four bulk-bulk and two bulk-boundary propagators for ϕ in terms of the mode functions

$$f(k,\eta) = \frac{H}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta} \,,$$

and compute $f' = \partial_\eta f$.

(iii) Compute the right-right and left-left contributions $B_{4,s}^{(rr+ll)}$ and show they take the form

$$B_{4,s}^{(rr+ll)} = \frac{C_0 s}{k_1 k_2 k_3 k_4} \sum_{a=1}^{3} \frac{C_a}{E_T^{\alpha_a}} \left(\frac{1}{E_R^{\beta_a}} + \frac{1}{E_L^{\beta_a}} \right) ,$$

where C_0, C_a, α_a and β_a are constants you should determine and

$$E_T = \sum_{a=1}^{4} k_a \,, \qquad k_I = |\mathbf{k}_1 + \mathbf{k}_2| \,, \qquad E_L = k_1 + k_2 + k_I \,, \qquad E_R = k_3 + k_4 + k_I \,.$$

(iv) Compute the left-right and right-left contributions $B_{4,s}^{(lr+rl)}$ and show they take the form

$$B_{4,s}^{(lr+rl)} = \frac{C_4 s}{k_1 k_2 k_3 k_4} \frac{1}{E_T^{\alpha_4}} \frac{1}{(E_R E_L)^{\beta_4}} \,,$$

where C_4, α_4 and β_4 are constants you should determine.

(v) State the scaling of $B_{4,s}$ that is expected from scale invariance and check that it is satisfied in your result.

2

Consider the following action for a scalar ϕ written in terms of ADM variables

$$S = \int d^4x\sqrt{h}N\left[\frac{M_{\rm Pl}^2}{2}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) + P(X,\phi)\right],$$

where $X = -\frac{1}{2}\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi$, and

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - {}^{(3)}\nabla_i N_j - {}^{(3)}\nabla_j N_i\right), \qquad g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2h^{ij} - N^iN^j \end{pmatrix},$$

where $\dot{h}_{ij} = \partial_t h_{ij}$.

(i) Write X in terms of ADM variables.

(ii) Vary the action with respect to N^i to find the momentum constraint

$$^{(3)}\nabla_j[K_i^j - \delta_i^j K] + \frac{P_{,X}}{M_{\rm Pl}^2 N}\partial_i\phi\left(N^j\partial_j\phi - \dot{\phi}\right) = 0\,.$$

(iii) Working to linear order in perturbations, $\phi(t,\mathbf{x}) = \bar{\phi}(t) + \varphi(t,\mathbf{x})$ and in flat gauge, we have

$$^{(3)}R \simeq 0\,, \qquad N \simeq 1 + \delta N\,, \qquad N_i \simeq a^2\partial_i\psi\,,$$
$$NK_{ij} \simeq a^2\left(H\delta_{ij} - \partial_i\partial_j\psi\right), \qquad NK \simeq 3H - \partial_i\partial_i\psi\,.$$

Solve the momentum constraint to find $\delta N = C\varphi$ where C is a background quantity you should determine.

(iv) Using the background Friedmann equations

$$3H^2M_{\rm Pl}^2 = 2XP_{,X} - P\,, \qquad -\dot{H}M_{\rm Pl}^2 = XP_{,X}\,,$$

show that $C = H\epsilon/\dot{\bar{\phi}}$, where $\epsilon = -\dot{H}/H^2$ is the slow-roll parameter.

(v) Using the results above, compute the tree-level bispectrum of φ induced by the cubic interaction

$$S_{\rm int} = -\int d^3x dt a^3\delta N\dot{\varphi}^2\,.$$

3

Consider a system of non-interacting dark matter particles of mass m each obeying

$$\mathbf{p}' = -am\nabla\phi\,, \quad \mathbf{p} = am\mathbf{x}'\,. \tag{1}$$

(i) Write down the Vlasov equation.

(ii) Using the following definitions

$$\rho \equiv \frac{m}{a^3}\int d^3p\, f\,, \qquad v^i \equiv \frac{\int d^3p\, \frac{p^i}{am} f}{\int d^3p\, f}\,, \qquad \sigma^{ij} \equiv \frac{\int d^3p\, \frac{p^i}{am}\frac{p^j}{am}\, f}{\int d^3p\, f} - v^i v^j\,, \tag{2}$$

derive the continuity and Euler equations.

(iii) Now consider standard perturbation theory:

(a) Draw the five diagrams contributing to the three-point function B_3 of the velocity divergence θ up to and including sixth order in $\delta^{(1)}$, namely at tree level and one loop.

(b) For each diagram write down the associated contribution to B_3. For each diagram, you need to consider only a single labelling of external momenta, rather than specifying all possible permutations.

(c) Given what you know about the renormalization of the matter power spectrum, what counterterm do you expect to cancel the UV-dependence of the loop integral corresponding to $\langle\theta^{(3)}\theta^{(2)}\theta^{(1)}\rangle$?

4

In this question you will derive the line-of-sight solution for free streaming photons, i.e. in the absence of collisions. You will work in flat gauge,

$$g_{00} = -a^2(1 + 2\delta N)\,, \qquad g_{0i} = a^2\partial_i\psi\,, \qquad g_{ij} = a^2\delta_{ij}\,,$$

and up to linear order in δN and ψ.

(i) Let P^μ be the 4-momentum of a photon and let $P^i = \frac{\epsilon}{a^2}\hat{p}^i$ with $\hat{p}^i\delta_{ij}\hat{p}^j = 1$. Derive P^0 from the on-shell relation.

(ii) Use the 0-component of the geodesic equation for photon to derive an equation satisfied by ϵ. You may use the following explicit expressions for the Christoffel symbols,

$$\Gamma^0_{00} = \mathcal{H} + \delta N'\,, \qquad \Gamma^0_{i0} = \partial_i\delta N + \mathcal{H}\partial_i\psi\,, \qquad \Gamma^0_{ij} = \mathcal{H}\,(1 - 2\delta N)\,\delta_{ij} - \partial_i\partial_j\psi\,,$$

where $\delta N' = \partial_\eta\delta N$ with η conformal time.

(iii) Write down the collision-less Boltzmann equation for photons. Parameterizing the phase-space density perturbations as

$$f(\mathbf{x},\mathbf{p},\eta) = \bar{f}\left[1 - \Theta(\mathbf{x},\mathbf{p},\eta)\frac{\partial\ln\bar{f}}{\partial\ln\epsilon}\right]\,,$$

expand the Boltzmann equation to linear order in Θ, δN and ψ.

(iv) Write down the line-of-sight solution of the linearized collision-less Boltzmann equation.

END OF PAPER