UNIVERSITY OF CAMBRIDGE

MAMA/311, NST3AS/311, MAAS/311

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 1:30 pm to 4:30 pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The following line element is a solution of the vacuum Einstein equation in six spacetime dimensions

$$\mathrm{d}s^2 = -\left(1 - \frac{r_+^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r_+^2}{r^2}} + r^2\mathrm{d}\Omega_3^2 + \mathrm{d}z^2,$$

where z is a periodic coordinate of period L, *i.e.* $z \sim z + L$, and

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

is the metric on a unit-radius round three-sphere with $\chi \in [0,\pi]$, $\theta \in [0,\pi]$, $\phi \sim \phi + 2\pi$. You may assume that $r_+ > 0$ and that we choose a time-orientation where $\partial/\partial t$ is future-directed for $r > r_+$.

(a) Show that the geodesic equation for null, spacelike and timelike geodesics can be reduced to an equation of the form

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \tilde{V}(r) = 0\,,$$

where τ is a suitable affine parameter. Determine $\tilde{V}(r)$. [10]

(b) Show that one can define a quantity $r_\star$ such that $u = t - r_\star$ and $v = t + r_\star$ are constant on radial outgoing and ingoing null geodesics, respectively. [5]

(c) Show that $r = r_+$ is a null hypersurface and is a Killing horizon of a Killing vector field K that you should determine, and find its associated surface gravity. What is the topology of this null hypersurface? [5]

(d) Using $(v, r, \chi, \theta, \phi, z)$ coordinates, show that the region $r < r_+$ is part of a black hole region. [10]

2 Consider a null geodesic congruence with tangent vector U in a four-dimensional spacetime $(\mathcal{M}, g)$.

(a) Define the *expansion* θ, *rotation* $\hat{\omega}$ and *shear* $\hat{\sigma}$ of a null geodesic congruence in terms of $B_{ab} = \nabla_b U_a$. [4]

(b) Show that

$$U^c \nabla_c B_{ab} + B^c{}_b B_{ac} = R_{cba}{}^d U_d U^c \,,$$

and thus determine $U^c \nabla_c \hat{\omega}_{ab}$ in terms of θ, $\hat{\sigma}$ and $\hat{\omega}$. [10]

(c) In this part of the question, assume that the null congruence contains the generators of a null hypersurface $\mathcal{N}$ and that we are only interested in the behaviour of those generators.

(i) Near $\mathcal{N}$, explain how to construct *Gaussian null* coordinates (λ, r, y^i):

$$\mathrm{d}s^2 = 2\mathrm{d}r\mathrm{d}\lambda + r^2 F \mathrm{d}\lambda^2 + 2r h_i \mathrm{d}\lambda \mathrm{d}y^i + h_{ij} \mathrm{d}y^i \mathrm{d}y^j \,,$$

with $i = 1, 2$. [8]

(ii) Show that

$$\frac{\partial \sqrt{h}}{\partial \lambda} = \theta \sqrt{h}$$

where $h = \det h_{ij}$ and give a physical interpretation for θ. [8]

3 State what an *asymptotically flat spacetime* at future null infinity $\mathcal{I}^+$ is. Explain in detail how this definition leads to a class of spacetimes that approach Minkowski spacetime near $\mathcal{I}^+$. You may assume in your answer that the spacetime $(\mathcal{M}, g)$ satisfies the vacuum Einstein equation. Additionally, you may use, without proof, that if $\bar{g} = \Omega^2 g$, then

$$R_{ab} = \bar{R}_{ab} + 2\Omega^{-1} \bar{\nabla}_a \bar{\nabla}_b \Omega + \bar{g}_{ab} \bar{g}^{cd} (\Omega^{-1} \bar{\nabla}_c \bar{\nabla}_d \Omega - 3\Omega^{-2} \bar{\nabla}_c \Omega \bar{\nabla}_d \Omega) \,,$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, $\bar{R}_{ab}$ are the components of the Ricci tensor of $\bar{g}$, and R_{ab} are the components of the Ricci tensor of g. [30]

4

(a) State the second law of black hole mechanics and sketch a proof. You may assume that the generators of the future horizon are complete to the future. [10]

(b) Consider a minimally coupled massless Klein-Gordon scalar field ϕ propagating on a fixed $D-$dimensional spacetime with coordinates $(\eta, z, \mathbf{x}) = (\eta, z, x^1, \ldots, x^{D-2})$ and metric

$$\mathrm{d}s^2 = a(\eta)^2\left(-\mathrm{d}\eta^2 + \mathrm{d}z^2\right) + \delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,, \quad \text{with} \quad i,j = 1,\ldots,D-2\,.$$

(i) Let

$$\phi(\eta, z, \mathbf{x}) = \int_{\mathbb{R}^{D-2}} \mathrm{d}\mathbf{k}^{D-2}\int_{-\infty}^{+\infty} \mathrm{d}k_z\,\Phi_{k_z\,\mathbf{k}}(\eta) e^{-\mathrm{i}k_z\,z - \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,.$$

Show that the scalar wave equation for $\phi(\eta, z, \mathbf{x})$ reduces to

$$\frac{\partial^2\Phi_{k_z\,\mathbf{k}}(\eta)}{\partial\eta^2} + \omega^2_{k_z\,\mathbf{k}}(\eta)\Phi_{k_z\,\mathbf{k}}(\eta) = 0,$$

for some function $\omega_{k_z\,\mathbf{k}}(\eta)$ that you should identify. [6]

(ii) Assume that $a(\eta)$ takes the constant values a_+ for $\eta > 0$ and a_- for $\eta < 0$. Furthermore, let M_- and M_+ denote the regions where $\eta < 0$ and $\eta > 0$, respectively. Assuming that the basis functions are C^1 everywhere, obtain the normalised positive frequency modes of $\Phi_{k_z\,\mathbf{k}}$ in $M_\pm$. [7]

(iii) Assume $\Phi_{k_z\,\mathbf{k}}$ is in the vacuum state in M_-. What is the expected number of particles with wavenumber $(k_z, \mathbf{k})$ in M_+? [7]

END OF PAPER