UNIVERSITY OF CAMBRIDGE

MAMA/310, NST3AS/310, MAAS/310

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 1:30 pm to 4:30 pm

PAPER 310

COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) From the continuity equation, determine the scaling of the energy density ρ with scale factor a for a component with constant equation of state parameter w. Hence show that the Hubble parameter can be written as $H(z) = H_0 E(z)$, with

$$E(z) = \left[\sum_i \Omega_{i,0} (1+z)^{3(1+w_i)} \right]^{1/2} ,$$

where you should define $\Omega_{i,0}$ and where the sum is over components i with constant equation of state parameters w_i.

(b) Explain how the expansion of the universe, parametrized by $a(t)$, can be determined given $\Omega_{i,0}$ and w_i for all components. Deduce how the scale factor a evolves with time for a universe whose energy density is dominated by cosmic strings (which have an equation of state parameter $w = -1/3$.)

Throughout the remainder of this question, you should consider a positively curved universe with matter, described by the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_m}{3} - \frac{k}{a^2},$$

where ρ_m is the matter energy density and k is the curvature parameter.

(c) Show that there is a parametric solution describing the time evolution of the scale factor a in terms of a parameter θ

$$a = A(1 - \cos\theta)$$

$$t = B(\theta - \sin\theta),$$

where you should determine the constants A and B.

(d) Expand a and t in powers of θ and hence show that at early times

$$\frac{a}{t^{2/3}} = C \left(1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} + \cdots \right)$$

where you should determine the constant C.

(e) Assume that, although radiation does not influence the background expansion of this universe, other galaxies and the CMB radiation are still visible in telescopes. Briefly discuss how galaxies and the CMB will appear in observations as the universe recollapses, i.e. $\theta \to 2\pi$.

2 In this question you will discuss recombination and the CMB power spectrum.

(a) At high temperatures ($T > 1$eV), electrons, protons and neutral Hydrogen are in equilibrium due to reactions such as $e^- + p \leftrightarrow H + \gamma$. Show that the equilibrium number density n_H of neutral Hydrogen is given by

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}$$

where n_e is the free electron number density, $B_H = m_e + m_p - m_H$ is the binding energy of Hydrogen, and m_e, m_p, m_H are the electron, proton and neutral Hydrogen masses. You may assume charge neutrality of the universe.

[*Hint: you may assume that the equilibrium number density for non-relativistic particles is* $n_i^{\mathrm{eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$]

(b) Hence derive the Saha equation, which describes the recombination process in equilibrium:

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2}\eta \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T},$$

where $X_e \equiv \frac{n_e}{n_b}$ is the free electron fraction and $\eta \equiv \frac{n_b}{n_\gamma} \sim 10^{-9}$ is the baryon-photon ratio, with n_b the baryon number density. You may neglect all nuclei other than protons (so that $n_b \approx n_p + n_H$, with n_p the proton number density) and may assume charge neutrality of the universe. You may also assume that the equilibrium number density of photons is given by $n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$.

Define the recombination temperature T_{rec} as when X_e falls below $X_e = 0.1$. Evaluating the Saha equation gives $T_{rec} \approx 0.3eV \approx 3600K$. Why is this temperature much lower than the naive recombination temperature $T \approx B_H \approx 13.6$eV?

(c) Explain succinctly why the CMB temperature power spectrum shows an oscillatory pattern that approximately follows a squared cosine in angular multipole. [You may neglect the impact of baryons in your explanation.] In your explanation, note the importance of the sound horizon. If the oscillatory pattern had followed a squared sine (rather than a cosine), what would this have implied?

[*Hint: You may wish to recall that the equation governing the Sachs-Wolfe term* $S = \phi + \delta_r/4$ *is* $\ddot{S}(\mathbf{k}, \tau) + \frac{k^2}{3} S(\mathbf{k}, \tau) = 0$ *and that the CMB power spectrum is approximately proportional to the square of the Sachs-Wolfe transfer function, i.e. that* $\ell(\ell+1)C_\ell \propto T_S^2(k = \frac{\ell}{\chi_*}, \tau_*)$.]

(d) A model of new physics at early times is proposed that changes physics during and prior to recombination. In this model, the energy densities of all components prior to recombination are increased beyond standard LCDM values by a factor $\rho \to \lambda\rho$ and B_H is modified from its standard value by $B_H \to \mu B_H$ (here, λ and μ are constants); these changes do not persist after recombination. Can a suitable choice of λ and μ leave the appearance of the CMB power spectrum approximately unchanged? Briefly and qualitatively justify your answer.

3 In this question you will discuss the effect of massive neutrinos on the matter power spectrum. You may assume in all parts of the question that the universe's energy density is dominated by matter, that all relevant scales are sub-horizon, and that linear perturbation theory is sufficiently accurate for all calculations. Subscripts of m, c, b and ν indicate matter, cold dark matter (CDM), baryon and neutrino energy densities, respectively. ρ_i is the total energy density of component i, and $\bar{\rho}_i$ is the unperturbed background energy density of this component.

(a) Starting from the approximate evolution equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0,$$

where dots indicate derivatives with respect to coordinate time t, show that the matter fractional density contrast δ_m grows with the scale factor a as $\delta_m \propto a$.

(b) Neutrinos have small masses and become non-relativistic after a redshift z_ν (you may make the approximation that all three neutrino species become non-relativistic at the same redshift). Neutrinos then contribute to the matter energy density along with CDM and baryons, so that $\rho_m = \rho_c + \rho_b + \rho_\nu$. To more accurately calculate the growth of the matter fractional density contrast $\delta_m = \frac{\delta\rho_c+\delta\rho_b+\delta\rho_\nu}{\bar{\rho}_c+\bar{\rho}_b+\bar{\rho}_\nu}$ after z_ν, one can separately compute the evolution of the CDM-baryon fractional density contrast $\delta_{cb} \equiv \frac{\delta\rho_c+\delta\rho_b}{\bar{\rho}_c+\bar{\rho}_b}$ and the neutrino density contrast $\delta_\nu \equiv \frac{\delta\rho_\nu}{\bar{\rho}_\nu}$. Show first that

$$\delta_m = (1 - f_\nu)\delta_{cb} + f_\nu\delta_\nu,$$

where f_ν is a constant defined as $f_\nu \equiv \frac{\Omega_{\nu,0}}{\Omega_{m,0}}$.

(c) The evolution of δ_{cb} can be described with the equation

$$\ddot{\delta}_{cb} + 2H\dot{\delta}_{cb} - 4\pi G\bar{\rho}_m\delta_m = 0.$$

Assume that $\delta_\nu \approx 0$ on the scales relevant to this question (because free-streaming rapidly smooths out variations in neutrino density). Hence show that the growth of δ_{cb} after z_ν is slightly slowed by the presence of massive neutrinos, with the growth given by:

$$\delta_{cb} \propto a^{X(f_\nu)}.$$

Here $X(f_\nu)$ is a function of f_ν that you should specify. You may assume that $f_\nu \ll 1$ so that all calculations can be performed to first order in f_ν.

(d) Deduce that when neutrinos have a non-negligible mass so that $f_\nu > 0$, the matter power spectrum $P_{f_\nu}(k, z)$ is suppressed compared to the matter power spectrum with massless neutrinos $P_{f_\nu=0}(k, z)$, with the suppression described by the expression

$$P_{f_\nu}(k, z) \approx (1 - 2f_\nu - \frac{6}{5}f_\nu \ln\frac{1+z_\nu}{1+z})P_{f_\nu=0}(k, z)$$

to first order in f_ν.

4

In this question you will discuss different models of the early universe.

(a) Consider a standard single-field slow-roll inflation model, where ϕ is the inflaton field and $V(\phi)$ is its potential; the Lagrangian for the inflaton is $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)$ and the equation of motion for the inflaton field is $\phi'' + 2\frac{a'}{a}\phi' - \nabla^2\phi + a^2\frac{\partial}{\partial\phi}V(\phi) = 0$, where superscript dots and primes indicate derivatives with time and conformal time, respectively. You may assume that, during inflation, $a(\tau) = -(H\tau)^{-1}$ (with τ the conformal time) and that $H = \sqrt{\frac{V(\phi)}{3M_{\rm pl}^2}} \approx$ constant. Canonical quantization leads to the following expression for the field operator $\hat{f} = a\hat{\delta\phi}$, describing perturbations to the inflation field $\delta\phi$:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[f^*_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} + f_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right]$$

$$\hat{\pi}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[(f^*)'_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} + f'_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right],$$

where $f_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$ and $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}$ are lowering and raising operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}'}$. By calculating the two point correlation function of $\delta\phi$, show that the dimensionless power spectrum of $\delta\phi$ after horizon exit is

$$\Delta^2_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2$$

[*Hint: you may assume that the dimensionless power spectrum $\Delta^2_{\delta\phi}$ is related to the two point correlation function via* $\langle 0|\hat{\delta\phi}(\tau, \mathbf{x})\hat{\delta\phi}(\tau, \mathbf{x}+\mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{2\pi^2}{k^3}\Delta^2_{\delta\phi}e^{-i\mathbf{k}\cdot\mathbf{r}}$.]

(b) Now assume that, in addition to the inflaton field ϕ, an additional scalar field σ, known as the "curvaton", is present in the early universe. This field's Lagrangian is $\mathcal{L} = \frac{1}{2}\dot{\sigma}^2 - \frac{1}{2a^2}(\nabla\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2$ and its equation of motion is $\sigma'' + 2\frac{a'}{a}\sigma' - \nabla^2\sigma + a^2m_\sigma^2\sigma = 0$; the curvaton has no interactions with the inflaton field. Assume that the energy density in the curvaton field is negligible at early times during inflation and that $m_\sigma \ll H$.

Consider the perturbed curvaton field $\sigma(\tau, \mathbf{x}) = \bar{\sigma}(\tau) + \frac{f^\sigma(\tau,\mathbf{x})}{a(\tau)}$, where $\bar{\sigma}$ is the background field value and $\delta\sigma \equiv f^\sigma/a$ is the perturbation in the field. Neglecting metric perturbations, show that each Fourier mode of the curvaton perturbation obeys the following equation: $(f^\sigma_{\mathbf{k}})'' + \left(k^2 - \frac{a''}{a}\right)f^\sigma_{\mathbf{k}} = 0$. Deduce the dimensionless power spectrum of fluctuations in the curvaton field, $\Delta^2_{\delta\sigma}(k)$, after horizon exit.

(c) After inflation ends, the inflaton field decays entirely into photons, reheating the universe. The background curvaton field subsequently evolves according to the Klein-Gordon equation $\ddot{\bar{\sigma}} + 3H\dot{\bar{\sigma}} + m_\sigma^2\bar{\sigma} = 0$. Assuming that the $3H\dot{\bar{\sigma}}$ term is negligible at this stage, show that the background curvaton field has an oscillatory solution. Average the density and pressure of the background curvaton field over many periods of oscillation and hence argue that the average energy density in the curvaton field falls as $\rho_\sigma \propto a^{-3}$.

[*Hint: you may assume that the energy density and pressure of a scalar field X are given by $\rho = \frac{1}{2}\dot{X}^2 + V$ and $P = \frac{1}{2}\dot{X}^2 - V$.*]

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Why is only the curvaton energy density relevant at late times? Assume that the curvaton decays into standard model particles at such late times. Could a cosmology in which all perturbations are generated by this curvaton field (rather than an inflaton field, which drives inflation) be consistent with our observations? Briefly justify your answer.

END OF PAPER