UNIVERSITY OF CAMBRIDGE

MAMA/309, NST3AS/309, MAAS/309

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The anti-de Sitter spacetime is the manifold $\mathbb{R}^4 = \{(t, \mathbf{x}) : t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3\}$ equipped with the metric

$$g = -\left(1 + |\mathbf{x}|^2\right) dt^2 + |d\mathbf{x}|^2 - \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 + |\mathbf{x}|^2}$$

a) The standard polar coordinates (r, θ, ϕ) are defined for $\mathbf{x} \neq 0$ by

$$\mathbf{x} = (r \cos\phi \sin\theta, r \sin\phi \sin\theta, r \cos\theta).$$

Show that the metric in the coordinates (t, r, θ, ϕ) takes the form

$$g = -w(r)dt^2 + \frac{dr^2}{w(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for some positive function $w(r)$ that you should determine.

b) Write down an action whose critical curves are the affinely parameterised geodesics of g in the (t, r, θ, ϕ) coordinates and use it to determine the Christoffel symbols of the metric in these coordinates.

c) Show that a geodesic which is initially moving radially (ie with $\dot{\theta} = \dot{\phi} = 0$) will remain radial, and that it satisfies

$$k = -\frac{e^2}{1 + r^2} + \frac{\dot{r}^2}{1 + r^2}$$

where k and e are constants whose significance you should comment on.

[Here $\dot{} = \frac{d}{d\lambda}$, where λ is the affine parameter of the geodesic]

d) Show that every radial timelike geodesic starting from $r = 0$ returns to $r = 0$ after a proper time T (as measured along the geodesic) that you should determine.

e) Show that every radial null geodesic starting from $r = 0$ at $t = 0$ has $r(\lambda) \to \infty$ and $t(\lambda) \to \tau$ as $\lambda \to \infty$ for some finite τ that you should determine.

2 Suppose that (M, g) is a smooth $(n+1)$-dimensional Lorentzian manifold. Let ∇ be the Levi-Civita connection.

a) i) Define the Riemann tensor $R^a{}_{bcd}$. You should justify carefully why any expression you give defines a tensor.

ii) Show that in a coordinate basis

$$R^\tau{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\sigma{}^\tau{}_\nu - \partial_\nu \Gamma_\sigma{}^\tau{}_\mu + \Gamma_\sigma{}^\rho{}_\nu \Gamma_\rho{}^\tau{}_\mu - \Gamma_\sigma{}^\rho{}_\mu \Gamma_\rho{}^\tau{}_\nu.$$

iii) Establish the Bianchi identities

$$R^a{}_{[bcd]} = 0, \qquad R^a{}_{b[cd;e]} = 0,$$

and the contracted Bianchi identity $R^a{}_{b;a} - \frac{1}{2}R_{;b} = 0$.
[You may assume the existence of normal coordinates about any point $p \in M$.]

b) The Bianchi identities imply the following identity for the Riemann tensor

$$\begin{aligned} 0 = {} & R_{abcd;e}{}^e + \alpha R_a{}^e{}_c{}^f R_{bedf} - 2R_a{}^e{}_d{}^f R_{becf} + R_{abef}R_{cd}{}^{ef} \\ & + R_{abec}R^e{}_d - R_{abed}R^e{}_c + R_{ad;bc} + R_{cb;ad} - R_{bd;ac} - R_{ac;bd}, \end{aligned} \qquad (*)$$

where α is a constant.

i) By considering an appropriate symmetry of the Riemann tensor determine α.

ii) Show that if g satisfies the vacuum Einstein equations then the Penrose wave equation holds:

$$0 = R_{abcd;e}{}^e + \alpha R_a{}^e{}_c{}^f R_{bedf} - 2R_a{}^e{}_d{}^f R_{becf} + R_{abef}R_{cd}{}^{ef}.$$

c) Suppose that the spacetime metric may be written in coordinates as a perturbation of the Minkoswki metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

for ϵ a small quantity. Formally expanding the Riemann tensor in ϵ as

$$R_{\mu\nu\sigma\tau} = R^{(0)}_{\mu\nu\sigma\tau} + \epsilon R^{(1)}_{\mu\nu\sigma\tau} + O(\epsilon^2)$$

explain why $R^{(0)}_{\mu\nu\sigma\tau}$ must vanish and use the result of the previous part to find an equation satisfied by $R^{(1)}_{\mu\nu\sigma\tau}$. Does your equation depend on a choice of gauge for the metric perturbation $h_{\mu\nu}$?

3

a) Suppose that a spacetime metric may be written in wave coordinates as a perturbation of the Minkoswki metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

Writing the energy-momentum tensor as $\epsilon T_{\mu\nu}$ and expanding to $O(\epsilon)$, derive the linearized Einstein equations in wave gauge

$$\partial^\rho \partial_\rho \overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \qquad \partial_\mu \overline{h}^\mu{}_\nu = 0, \tag{$*$}$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h_\tau{}^\tau \eta_{\mu\nu}$, and indices are raised and lowered with the Minkowski metric.

You may assume that in any coordinate basis the Ricci tensor may be written

$$\begin{aligned} R_{\sigma\nu} = &-\frac{1}{2} g^{\mu\rho} \partial_\mu \partial_\rho g_{\sigma\nu} + \Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu}\Gamma^\tau{}_\sigma{}^\lambda + \Gamma_{\lambda\tau\sigma}\Gamma^\tau{}_\nu{}^\lambda \\ &+ \frac{1}{2}\partial_\sigma \Gamma_{\mu\nu}{}^\mu + \frac{1}{2}\partial_\nu \Gamma_{\mu\sigma}{}^\mu - \Gamma_{\mu\lambda}{}^\mu \Gamma_\nu{}^\lambda{}_\sigma \end{aligned}$$

and that the wave coordinate condition takes the form $\Gamma_\mu{}^{\nu\mu} = 0$.

b) Suppose $h_{\mu\nu}$ solves $(*)$. A gauge transformation generated by a vector field ξ^μ acts on $h_{\mu\nu}$ by

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$$

Show that $h'_{\mu\nu}$ will also solve $(*)$ if $\partial^\rho \partial_\rho \xi^\mu = 0$.

c) Consider a perturbation of the form

$$\overline{h}_{\mu\nu} = H_{\mu\nu}(x^\sigma k_\sigma)$$

where k^σ is a constant 4−vector and $H_{\mu\nu}(s) = H_{\nu\mu}(s)$ for $\mu, \nu = 0, \ldots, 3$ are functions of a single real variable which decay as $|s| \to \infty$.

i) Find conditions on $k^\sigma, H_{\mu\nu}(s)$ such that $\overline{h}_{\mu\nu}$ solves $(*)$ for $T_{\mu\nu} = 0$.

ii) Show that by making a choice of coordinate axes, together with appropriately chosen gauge transformations the solution can be brought to the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(z-t) & f_\times(z-t) & 0 \\ 0 & f_\times(z-t) & -f_+(z-t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $f_+, f_\times$ are arbitrary functions and $x^\mu = (t, x, y, z)$.

[Hint: You may wish to consider a gauge transformation generated by the vector field $\xi^0 = 0$, $\xi^i = \int_{-\infty}^{z-t} H_0{}^i(s)ds$, *followed by a transformation generated by* $\xi^0 = -\xi^3 = \alpha \int_{-\infty}^{z-t} H_\mu{}^\mu(s)ds$, $\xi^1 = \xi^2 = 0$, *for* α *a suitable constant.]*

4 Let (M, g) be a 4−dimensional Lorentzian manifold, and ∇ the Levi-Civita connection.

a) Let ω be a p-form.

i) Working in a coordinate chart define the exterior derivative $d\omega$ and show that your definition does not depend on the choice of chart.

ii) Show that the exterior derivative satisfies $d(d\omega) = 0$ and $d(\phi^*\omega) = \phi^* d\omega$, where ϕ is a diffeomorphism.

iii) For a vector field X which generates a smooth one-parameter family of diffeomorphisms ϕ_t^X, define the Lie derivative $\mathcal{L}_X\omega$ in terms of ϕ_t^X. Deduce that $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$.

b) Let $T_{ab}{}^{cd} = \epsilon_{ab}{}^{cd}$, where ϵ_{abcd} is the volume form of the spacetime. Show that $(\mathcal{L}_X T)_{ab}{}^{cd} = 0$ if and only if X^a satisfies the *conformal Killing equation*:

$$\nabla_a X_b + \nabla_b X_a = \alpha g_{ab} \nabla_c X^c$$

where α is a constant that you should determine.

You may find the following identities helpful

$$(\mathcal{L}_X T)_{ab}{}^{cd} = X^e \nabla_e T_{ab}{}^{cd} + T_{eb}{}^{cd} \nabla_a X^e + T_{ae}{}^{cd} \nabla_b X^e - T_{ab}{}^{ed} \nabla_e X^c - T_{ab}{}^{ce} \nabla_e X^d$$

$$\nabla_a \epsilon_{bcde} = 0, \qquad \epsilon_a{}^{pqr} \epsilon_{bpqr} = -6 g_{ab}, \qquad \epsilon_{ab}{}^{pq} \epsilon_{cdpq} = -2(g_{ac} g_{bd} - g_{ad} g_{bc}).$$

c) Recall that the vacuum Maxwell equations for a 2−form F are

$$dF = 0, \qquad d(\star F) = 0$$

where $(\star F)_{ab} = \frac{1}{2} \epsilon_{ab}{}^{cd} F_{cd}$. Show that if X satisfies the conformal Killing equation then $\tilde{F} = \mathcal{L}_X F$ satisfies the vacuum Maxwell equations whenever F does.

END OF PAPER