UNIVERSITY OF CAMBRIDGE

MAMA/307, NST3AS/307, MAAS/307

MAT3

MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 9:00 am to 11:00 am

PAPER 307

SUPERSYMMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Cover sheet	None
Treasury tag	
Script paper	
Rough paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This question concerns renormalisable $N = 1$ globally supersymmetric $SU(3)$ gauge theory with a single chiral superfield Φ in the fundamental representation of $SU(3)$ and with a vector superfield $V = V_a T_a$ where T_a are generators of $SU(3)$ in the fundamental representation. Strictly speaking, this theory possesses a gauge anomaly, but we shall ignore it here. The supersymmetry generators Q_α, $\overline{Q}_{\dot{\alpha}}$ have representations $\mathcal{Q}_\alpha$, $\overline{\mathcal{Q}_{\dot{\alpha}}}$ when acting on a function on superspace and we shall denote SUSY covariant derivatives by $\mathcal{D}_\alpha$, $\overline{\mathcal{D}}_{\dot{\alpha}}$.

(i) Under a SUSY transformation, a general superfield S transforms as

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \to \exp[i(\epsilon\mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})]S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}),$$

where ϵ^α and $\bar{\epsilon}^{\dot{\alpha}}$ are Grassman-valued SUSY transformation parameters. From this, find the explicit forms of $\mathcal{Q}_\alpha$, $\overline{\mathcal{Q}_{\dot{\alpha}}}$ as differential operators. You may assume any properties of the super-Poincaré algebra from lectures, provided that they are clearly stated.

(ii) What is the defining property of a *vector superfield* V_a?

(iii) What is the defining property of a *chiral superfield* Φ?

(iv) A generalised $SU(3)$ gauge transformation is given by $\Phi \to \exp(-2i\Lambda)\Phi$, where $\Lambda = \Lambda_a T_a$. Give the mass dimension of the superfields Λ_a and state clearly any restrictions on on their nature.

The SUSY $SU(3)$ field-strength chiral superfield is

$$W_\alpha = -\frac{1}{8}\overline{\mathcal{D}}^2\left(\exp(-2V)\mathcal{D}_\alpha \exp(2V)\right).$$

(v) Write down the theory's Lagrangian density in terms of superspace integrals involving functions of V_a, Φ and W_α.

(vi) Assuming generalised $SU(3)$ gauge invariance of the standard Kähler potential, relate V and V' where $V \to V' = V'_a T_a$ under a finite generalised gauge transformation.

(vii) Using the relation you derived in (vi), show that the entire Lagrangian density is invariant under a generalised gauge transformation.

2 Describe why theoretical arguments lead us to expect that the order of magnitude of sparticle masses is a few TeV or less. Your answer should include Feynman diagrams and order of magnitude estimates.

Define $R-$parity in terms of spin s, baryon number B and lepton number L.

Write down the $R-$parity violating superpotential of the MSSM W_{RPV} [you may leave gauge indices suppressed].

Draw a Feynman diagram to illustrate that two of the baryon/lepton number violating terms in W_{RPV} can induce proton decay into a pion and an invisible particle. Is the invisible particle a neutrino or an anti-neutrino? Which family can it be from? What is the electric charge of the pion?

By dimensional arguments, estimate the rough order of magnitude of the decay rate of the proton via this channel in terms of the Yukawa coupling constants in W_{RPV} assuming that SUSY particle masses are 10 TeV.

The experimental lower bound on the proton lifetime through this channel is approximately

$$\tau_{p\rightarrow\mu^+\pi^0} > 10^{34} \text{ years.}$$

Use this and dimensional arguments to determine an order of magnitude upper bound on the product of the two Yukawa couplings that give rise to proton decay above.

Show that R-parity forbids all of the terms in W_{RPV}. Now impose $R-$parity on the theory.

In the MSSM Lagrangian, there are direct couplings between a (non-gauge) particle, its anti-particle, an $SU(2)_L$ $W-$boson and a gluon. Which part of the Lagrangian density does this term come from? Draw a Feynman diagram for it and write its Feynman rule.

[*In natural units, 1 second is of order* 10^{24} GeV^{-1} *and the mass of a proton is 1* GeV.]

3 The generators of the Poincaré group $M^{\mu\nu}$ and P^{σ} satisfy

$$[M^{\mu\nu}, P^{\sigma}] = i(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma}).$$

Define the Pauli-Lubjanski operator W_{μ} in terms of $M^{\mu\nu}$ and P^{μ}.

Write down a general Poincaré transformation on a left-handed spinor field $\psi_{\alpha}(x)$ in terms of the left-handed spinor representation of the Lorentz group algebra

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right),$$

which satisfies

$$\sigma^{\mu\nu} = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}.$$

In terms of $(SU(2)_L, SU(2)_R)$ representation spaces of the Lorentz algebra, $\psi \sim (\frac{1}{2}, 0)$. Write down $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$ in terms of irreducible $(SU(2)_L, SU(2)_R)$ representation spaces. Show this explicitly by the decomposition of $\psi_{\alpha}\chi_{\beta}$, where $\chi \sim (\frac{1}{2}, 0)$.

Write down all of the commutators or anti-commutators of the super-Poincaré algebra which involve the supersymmetry generators Q_{α} and/or $\bar{Q}^{\dot{\alpha}}$.

(i) Defining $B_{\mu} = W_{\mu} - \frac{1}{4}(\bar{Q}\bar{\sigma}_{\mu}Q)$, calculate $[B_{\mu}, P_{\rho}]$.

(ii) Defining $C_{\mu\nu} = B_{\mu}P_{\nu} - B_{\nu}P_{\mu}$, calculate $[C_{\mu\nu}, P_{\rho}]$.

(iii) Compute $[\bar{Q}\bar{\sigma}_{\mu}Q,\, Q_{\alpha}]$.

(iv) Compute $[B_{\mu}, Q_{\alpha}]$.

(v) Compute $[C_{\mu\nu}, Q_{\alpha}]$.

(vi) Defining the Lorentz scalar superspin operator

$$\tilde{C}_2 = C_{\mu\nu}C^{\mu\nu},$$

write down the implied commutator with $M^{\mu\nu}$. Thus, demonstrate that $\tilde{C}_2$ is a Casimir of the super-Poincaré algebra.

END OF PAPER