UNIVERSITY OF CAMBRIDGE

MAMA/306, NST3AS/306, MAAS/306

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question concerns the classical motion of the bosonic string in $\mathbb{R}^{1,D-1}$. The Polyakov action for a string with embedding coordinate $X^\mu(\sigma,\tau)$ and world-sheet metric $g_{\alpha\beta}(\sigma,\tau)$ is,

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X,$$

where $T = 1/(2\pi\alpha')$ is the string tension.

(a) Derive the classical Virasoro constraint on string motion and the equation of motion for the string in conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$. Determine the possible boundary conditions for an open bosonic string.

(b) Show that the action is invariant under spacetime translations and Lorentz transformations and demonstrate the existence of corresponding conserved currents,

$$P^\alpha_\mu = T\partial^\alpha X_\mu \quad \text{and} \quad J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu.$$

Write down the corresponding conserved Noether charges P_μ and $J_{\mu\nu}$ for the open string.

(c) The open string with Neumann boundary conditions in all directions has a mode expansion,

$$X^\mu(\sigma,\tau) = x^\mu + \alpha' p^\mu\tau + i\alpha'\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\tau}\cos(n\sigma).$$

Evaluate the charges P_μ and $J_{\mu\nu}$ in terms of the variables x^μ, p^μ and α^μ_n with $n\in\mathbb{Z}$, $n\neq 0$.

(d) Now fix the residual gauge invariance by setting $X^0(\sigma,\tau) = A\tau$ for an undetermined constant A. Write down a solution for string motion corresponding to a stretched open string rigidly rotating in the plane with coordinates $x = X^1$ and $y = X^2$. You should check explicitly that your solution obeys the string equation of motion and the Virasoro constraint.

Determine the mass M with $M^2 = -P_\mu P^\mu$ and angular momentum $J = J_{12}$ of your solution in terms of A and T.

2

A free scalar field $X(z, \bar{z})$ in two dimensions has action,

$$S = \frac{1}{2\pi}\int d^2z\, \partial_z X \bar{\partial}_z X.$$

In this question we adopt the conventions from the lectures for complex coordinates $z = \sigma + i\rho$. In particular ∂_z and $\bar{\partial}_z$ denote partial derivatives with respect to z and $\bar{z}$ considered as independent variables and $d^2z = 2d\sigma d\rho$.

(a) Using either operator or path integral methods, calculate the Euclidean two-point function, $\langle \partial_z X(z) \partial_w X(w) \rangle$.

(b) A normal-ordered holomorphic stress-energy tensor for the theory is defined by,

$$\begin{aligned} T(z) &= -:\partial_z X(z)\partial_z X(z): \\ &= -\lim_{\delta\to 0}\left[\partial_z X\left(z+\frac{1}{2}\delta\right)\partial_z X\left(z-\frac{1}{2}\delta\right) - \left\langle \partial_z X\left(z+\frac{1}{2}\delta\right)\partial_z X\left(z-\frac{1}{2}\delta\right)\right\rangle\right]. \end{aligned}$$

Using Wick's theorem (which you need not prove), compute the singular terms in the product $T(z)\partial_w X(w)$.

(c) Consider a conformal transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ under which the scalar field transforms as, $X(z,\bar{z}) \to \tilde{X}(w,\bar{w}) = X(z,\bar{z})$ and let $w^{(n)}(z)$ denote the n'th derivative $d^n w/dz^n$. With reference to its properties under this transformation, explain what is meant by a *primary operator* of conformal weight $(h, \tilde{h})$. Give an example of a primary operator in this theory and state its conformal weight.

(d) Rewrite $T(z)$ in terms of the transformed stress-energy tensor,

$$\tilde{T}(w) = -:\partial_w \tilde{X}(w)\partial_w \tilde{X}(w):,$$

thereby showing that,

$$\tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2}[T(z) - R(w;z)]$$

for a function $R(w,z)$ depending on $w(z)$ and its derivatives $w^{(n)}(z)$ for $n = 1, 2, 3$ which you should determine. Is $T(z)$ a primary operator? Justify your answer.

3

(a) Starting from the Polyakov path integral, describe the derivation of the tree-level scattering amplitude for m tachyons of the closed bosonic string in the form,

$$\mathcal{A}^{(m,0)}(p_1, p_2, \ldots, p_m) = g_s^{m-2}\delta^{(26)}\left(\sum_{i=1}^{m} p_i\right)\int \frac{\prod_{i=1}^{m} d^2 z_i}{\mathrm{Vol}\,[SL(2,\mathbb{C})/\mathbb{Z}_2]}\prod_{j<l}|z_j - z_l|^{\alpha' p_j\cdot p_l}\,. \quad (*)$$

(b) Consider the hard scattering limit where $s_{ij} = -(p_i+p_j)^2 \to \infty$ with all ratios s_{ij}/s_{kl} held fixed. By writing the product appearing in integrand of $(*)$ as $\exp(f(\{z_i\},\{\bar{z}_i\}))$, for an appropriately chosen function f, and looking for stationary points where $\partial f/\partial z_i = \partial f/\partial \bar{z}_i = 0$ for $i = 1, 2, \ldots n$, obtain a set of algebraic equations for the insertion points z_i which determine the asymptotics of the scattering amplitude in this limit.

(c) For the special case $m = 4$, use $SL(2,\mathbb{C})$ invariance to fix the positions of z_1, z_2 and z_4 in $(*)$ to convenient values and solve the equations obtained in part b) to find the stationary value of $z = z_3$. Hence or otherwise show that,

$$\mathcal{A}^{(m,0)}(p_1, p_2, p_3, p_4) \quad\sim\quad g_s^{m-2}\delta^{(26)}\left(\sum_{i=1}^{m} p_i\right)\exp\left[-\frac{\alpha'}{2}\left(s\log s + t\log t + u\log u\; + \ldots\right)\right],$$

where $s = s_{34}$, $t = s_{13}$ and $u = s_{23}$ and the dots indicate terms in the exponent which grow slower than $s\log s$ in the hard scattering limit s, t, $u \to \infty$.

4

A two-dimensional non-linear σ-model for scalar fields corresponding to maps,

$$X : \Sigma \quad \to \quad \mathcal{M},$$

has action,

$$S \quad = \quad \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{g}g^{\alpha\beta}G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu,$$

where $g_{\alpha\beta}$ and $G_{\mu\nu}$ are Riemannian metrics on the two-dimensional spacetime Σ and the target space $\mathcal{M}$ respectively.

(a) Explain briefly the relevance of this model for string theory and state without proof necessary conditions for $\mathcal{M}$ to be a consistent background for the bosonic string.

(b) Consider the case where Σ is $\mathbb{R}^2$ with a flat metric and $\mathcal{M}$ is a two-sphere S^2 of radius R with the standard round metric,

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu \quad = \quad R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right),$$

where $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$ are the usual polar and azimuthal angles on the sphere respectively. Choosing a complex coordinate $Z = \tan(\theta/2)\exp(i\phi)$ on the sphere, show that the action can be put in the form,

$$S \quad = \quad \frac{1}{\lambda^2}\int_{\Sigma} d^2\sigma \frac{\partial_\alpha Z \partial^\alpha \bar{Z}}{\left(1 + Z\bar{Z}\right)^2},$$

where λ is a dimensionless coupling constant which you should determine.

(c) Formulate perturbation theory in powers of λ by expanding the field in fluctuations around the North Pole, at $Z = 0$, setting $Z = \lambda\delta Z$. In particular, you should give Feynman rules for the propagator of the fluctuation field δZ and for its leading interaction vertex. With brief justification, identify a divergent Feynman diagram which contributes to the renormalisation of the metric $G_{\mu\nu}$ and evaluate it in a regularisation scheme of your choice. Briefly discuss the implications of your result for string propagation on a spacetime of the form $\mathcal{M} = S^2 \times \mathcal{N}$ where $\mathcal{N}$ is arbitrary. [In this question, you are not expected to accurately determine the overall normalisation of the contribution you evaluate and you need not account for other possible renormalisations (eg wavefunction renormalisation).]

END OF PAPER