MAMA/305, NST3AS/305, MAAS/305

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 17 June 2025 9:00 am to 12:00 pm

PAPER 305

THE STANDARD MODEL

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Abelian Higgs model consists of a $U(1)$ gauge field A_μ coupled to a complex scalar field ϕ. The action is

$$S = \int d^4x \,\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi - \frac{\lambda}{2}(\phi^\dagger\phi - v^2)^2\Big).$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$.

(i) By working in a suitable gauge, determine the masses and spins of the particle excitations in this theory.

(ii) Show that the equation of motion for the gauge field takes the form

$$\partial_\mu F^{\mu\nu} = J^\nu$$

for some current J^ν that you should compute. Show that, when restricted to configurations such that $|\phi| = v$, the curl of the spatial current $\nabla \times \mathbf{J}$ is related to the magnetic field $\mathbf{B}$. Hence show that, in time-independent backgrounds, the magnetic field $\mathbf{B}$ obeys

$$\nabla^2\mathbf{B} = \frac{1}{\xi^2}\mathbf{B}$$

for some penetration depth ξ that you should compute. Explain the physical significance of this result for superconductivity.

(iii) Consider now a $U(1)$ gauge field coupled to two complex scalar fields ϕ_i, $i = 1, 2$, with action

$$S = \int d^4x \,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{2}\mathcal{D}_\mu\phi_i^\dagger\mathcal{D}^\mu\phi_i - \frac{\lambda}{2}\left(\sum_{i=1}^{2}\phi_i^\dagger\phi_i - v^2\right)^2\right).$$

What is the global symmetry of the theory?

Show that there is a combination of the gauge symmetry and global symmetry that is unbroken in the ground state. What is the spectrum of particle masses? Confirm that the number of massless scalars is in agreement with expectations from Goldstone's theorem. What is the manifold of gauge inequivalent ground states?

2 Consider QCD coupled to two flavours of quarks, up and down, described by Dirac fermions q_i with $i = 1, 2$. Take both quarks to be massless.

(i) Write down the Lagrangian for the theory. What is the global symmetry group of the classical theory? What is the global symmetry group of the quantum theory?

(ii) Assume that a quark condensate forms, given by

$$\langle \bar{q}_{L\,i} q_{R\,j} \rangle = -\sigma \delta_{ij}$$

where q_L and q_R are left- and right-handed fermions respectively, and σ is a dimensionful constant. Describe the resulting symmetry breaking. What is the vacuum manifold of this theory? Explain why this necessarily means that the low-energy physics, with no more than two derivatives, is described by the Lagrangian

$$\mathcal{L}_{\text{pion}} = \frac{f_\pi^2}{4} \text{tr} \, (\partial^\mu U^\dagger \partial_\mu U)$$

for a matrix-valued field $U(x) \in SU(2)$. What is the dimension of the constant f_π?

(iii) Write the field as $U(x) = \exp(2i\pi(x)/f_\pi)$ with the pion field $\pi(x) \in su(2)$, the Lie algebra of $SU(2)$. Show that the terms quadratic and quartic in π take the form

$$\mathcal{L}_{\text{pion}} = \text{tr}(\partial_\mu \pi)^2 - \frac{2}{3f_\pi^2} \text{tr}(\pi^2 (\partial_\mu \pi)^2 - (\pi \partial_\mu \pi)^2)$$

(iv) The pion fields can be coupled to the $U(1)$ of electromagnetism by introducing the charge matrix

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} .$$

The photon field A_μ is coupled to $U(x)$, with coupling constant e, by promoting the derivative to a covariant derivative

$$\mathcal{D}_\mu U = \partial_\mu U - ieA_\mu [Q, U] \ .$$

Explain, in terms of the symmetries of the theory, why this is the right coupling. By examining how the covariant derivative acts on the pion field, determine the electric charges of the various components of $\pi \in su(2)$.

3 Briefly describe the implications if a symmetry has:

- a gauge anomaly,
- a chiral, or ABJ, anomaly,
- a 't Hooft anomaly.

Consider $SU(2)$ Yang-Mills coupled to N_f massless Dirac fermions in the fundamental representation, decomposed into left- and right-handed Weyl fermions as

$$S_0 = \int d^4x \, \left(-\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + i \sum_{i=1}^{N_f} \bar{\psi}_{L\,i} \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L^i + \bar{\psi}_{R\,i} \sigma^\mu \mathcal{D}_\mu \psi_R^i \right) \, .$$

(i) The global symmetry group of the classical theory is $SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$. What is the global symmetry group of the quantum theory? Write a table showing how the fermions transform under this symmetry.

(ii) Compute the $SU(N_f)_L^3$, $SU(N_f)_L^2 U(1)$, $U(1)^3$ and $U(1)$-gravity 't Hooft anomalies, where the $U(1)$ is the appropriate global symmetry of the quantum theory. Explain why confinement necessarily implies chiral symmetry breaking in this theory.

(iii) Now consider the same theory, but with addition of a massless, left-handed Weyl fermion λ_L in the adjoint representation of the gauge group. All subsequent questions relate to this extended theory. Does the theory with the extra adjoint Weyl fermion suffer a gauge anomaly? If not, why not?

(iv) The theory has an additional global $U(1)$ symmetry that we call $U(1)_X$

$$U(1)_X: \quad \psi_L \to e^{-i\alpha} \psi_L \ , \quad \psi_R \to e^{+i\alpha} \psi_R \ , \quad \lambda_L \to e^{iq\alpha} \lambda_L \ .$$

For what value of q is this a symmetry of the quantum theory? [*You may use the Dynkin indices for* $SU(N)$*:* $I(\square) = I(\overline{\square}) = 1$ *and* $I(\mathrm{adj}) = 2N$.]

(v) Compute the 't Hooft anomalies for $SU(N_f)_L^3$, $SU(N_f)_L^2 U(1)_X$ and $U(1)_X^3$, as well as the mixed $U(1)_X$-gravity 't Hooft anomaly.

(vi) It is conjectured that, for $N_f = 3$, the theory with the adjoint fermion confines without breaking chiral symmetry, with the massless degrees of freedom given by the *left-handed* gauge singlets

$$\chi_i{}^j = \bar{\psi}_{R\,i} \lambda_L \psi_L^j \ , \quad \eta_i = \epsilon_{ijk} \psi_L^j \lambda_L \psi_L^k \ , \quad \tilde{\eta}^i = \epsilon^{ijk} \bar{\psi}_{R\,j} \lambda_L \bar{\psi}_{R\,k}$$

How do these transform under the global symmetry group, including $U(1)_X$?

(vii) Treating χ, η and $\tilde{\eta}$ as free fermions, compute the 't Hooft anomalies for $SU(N_f)_L^3$, $SU(N_f)_L^2 U(1)_X$, $U(1)_X^3$, and for $U(1)_X$-gravity. Show that they match the 't Hooft anomalies of the gauge theory.

4 List the fundamental fermions in one generation of the Standard Model, including the right-handed neutrino, and give their representations under the gauge group.

(i) Explain why it's not possible to write down mass terms for the fermions but how, with judiciously chosen quantum numbers for the Higgs boson, it is possible for them all to gain a mass through Yukawa terms.

(ii) If the Standard Model had N generations, how many physical parameters would there be in the quark Yukawa couplings? How does this counting split between mass parameters, mixing angles, and complex phases? What are the implications for time-reversal invariance?

(iii) The W-boson couples to the quarks through the current

$$\mathcal{L}_{\text{weak}} \sim (W_\mu^+ J_+^\mu + W_\mu^- J_-^\mu)$$

where

$$J_\mu^+ = \sum_{i=1}^{N} \bar{u}_L^i \bar{\sigma}_\mu d_L^i \quad \text{and} \quad J_\mu^- = \sum_{i=1}^{N} \bar{d}_L^i \bar{\sigma}_\mu u_L^i \ .$$

with u_L and d_L the left-handed up-like and down-like quarks respectively. Explain how, for $N = 2$, diagonalising the Yukawa matrices results in the Cabibbo angle θ_c. Write down the coupling between the W-bosons and fermions in the fermion mass eigenbasis. Describe a process that is only possible because $\theta_c \neq 0$.

END OF PAPER