UNIVERSITY OF CAMBRIDGE

MAMA/304, NST3AS/304, MAAS/304, NST3PHY/2/AQFT, MAPY/2/AQFT

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Massive ϕ^4 theory in four dimensional Minkowski spacetime has Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial \phi^\mu - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4.$$

(a) Draw all diagrams up to and including order λ^2 that contribute to the quadratic and quartic terms in the quantum effective action, $\Gamma_2(p)$ and $\Gamma_4(s,t,u)$ respectively, where s, t and u are the Mandelstam variables.

(b) Using a momentum cut-off Λ to regularise the theory, evaluate the one-loop contribution to Γ_4. By introducing a counter-term Lagrangian which includes

$$-\frac{\delta_\lambda}{4!}\phi^4,$$

show that the requirement that $\Gamma_4 = \lambda$ when all external momenta are zero gives

$$\delta_\lambda = -\frac{3\lambda^2}{32\pi^2} \ln\left(\frac{\Lambda^2}{F}\right),$$

as a suitable choice of counter-term, where F should be given explicitly.

(c) Hence find an expression for the renormalized $\Gamma_4(s,t,u)$ and show that, to leading order the beta function for the coupling λ is

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + ...$$

[*Hint: The following integral*

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2},$$

may be used without proof.]

2

A free massive scalar field in four-dimensional Minkowski spacetime has Lagrangian

$$\mathcal{L}_b = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2,$$

and generating functional

$$Z_0[J] = \mathcal{N}\int \mathcal{D}\phi\, \exp\left(i\int d^4x\Big(\mathcal{L} + J(x)\phi(x)\Big)\right),$$

where $\mathcal{N}$ is some appropriately chosen normalization constant and J is a source.

(a) With an appropriate choice of $\mathcal{N}$, which you should find, show that the generating functional may be written as

$$Z_0[J] = \exp\left(-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right),$$

where $D_F(x-y)$ should be given explicitly. Your expression for $\mathcal{N}$ may be written as a functional integral and need not be evaluated explicitly.

(b) Consider a massive spinor field with Lagrangian

$$\mathcal{L}_f = i\bar{\psi}(\not{\partial} - m)\psi.$$

Derive an expression for the free spinor generating functional $Z_0[\bar{\eta},\eta]$ where η and $\bar{\eta}$ are sources for the spinor fields $\bar{\psi}$ and ψ respectively. Your expression should include $S_F(x-y)$, which you should give an explicit expression for.

(c) Now consider a theory with Lagrangian $\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\rm int}$, where $\mathcal{L}_b$ and $\mathcal{L}_f$ are as above and

$$\mathcal{L}_{\rm int} = -g\bar{\psi}\phi\psi.$$

Write down an expression for the generating functional $Z[\bar{\eta},\eta,J]$ for this theory, where η and $\bar{\eta}$ are sources for the spinors and J a source for the scalars. Your expression should be written as an explicit functional of the sources and their derivatives and should not involve scalar and spinor fields directly.

(d) Now assume that the mass M is much larger than m and all momenta, i.e., $M^2 \gg m^2$ and $M^2 \gg p^2$. By directly evaluating $D_F(x-y)$ in this limit and expanding out the generating functional $Z[\bar{\eta},\eta,J]$ to order g^2, show that there is a four-spinor interaction in the low energy effective theory. Give an expression for the effective coupling constant for this four-spinor interaction to leading order in g and check that it has the correct mass dimension.

3

Consider the Lagrangian for the scalar field $\phi(x)$ in d-dimensional Minkowski space

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{n!}\lambda\phi^n + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_{m^2}\phi^2 - \frac{1}{n!}\delta_\lambda\phi^n,$$

where n is an integer.

(a) Briefly explain the role of each of the terms in the Lagrangian and, without derivation, write down the momentum space Feynman rules of this theory.

(b) Using dimensional regularizaton and working in the minimal subtraction scheme, determine δ_Z and δ_{m^2} at one-loop for the case where $n = 3$ and $d = 6$.

(c) Now let $n = 6$. For what value of d is the coupling λ marginal? Draw the diagram you would need to evaluate in order to determine the leading order contribution contribution to the two-point function for the $n = 6$ theory in this dimension (you do not need to evaluate the diagram). Does this diagram contribute to δ_Z? Explain your answer.

[*Hint: You may find the following integrals useful*

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}, \qquad \int_{\mathbb{R}^d} \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}}.$$

These integrals may be used without proof. The behaviour of the Gamma function near zero is

$$\Gamma(z) = \frac{1}{z} - \gamma + ...,$$

where γ is the Euler-Mascheroni constant.]

4

The Lagrangian for $SU(N)$ Yang-Mills theory coupled to scalars in the adjoint representation is given by

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}(D_\mu\phi D^\mu\phi) - \frac{1}{3!}\mathrm{Tr}(\phi^3), \qquad (\star)$$

where

$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu,\phi] \qquad \text{and} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

The fields A_μ and ϕ are traceless Hermitian matrices and transform under the gauge symmetry as

$$A_\mu \to UA_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger, \qquad \phi \to U\phi U^\dagger,$$

where $U \in SU(N)$.

(a) Show that $D_\mu\phi$ transforms covariantly under gauge transformations. Show that

$$[D_\mu, D_\nu]\phi = -ig[F_{\mu\nu},\phi],$$

and hence deduce that $F_{\mu\nu}$ transforms covariantly. Explain why the Lagrangian is gauge-invariant.

(b) We introduce generators T_a for the gauge group, where

$$[T_a, T_b] = if_{ab}{}^cT_c, \qquad \mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$$

where $f_{ab}{}^c$ are structure constants. The anti-commutator of two generators is given by

$$\{T_a, T_b\} = \frac{1}{N}\delta_{ab} + d_{ab}{}^cT_c,$$

where $d_{abc} = d_{ab}{}^e\delta_{ce}$ is constant and symmetric in all indices. Find an explicit expression for the cubic scalar potential in $(\star)$ in terms of $f_{ab}{}^c$, d_{abc} and the scalar fields ϕ^a, where $\phi = \phi^aT_a$.

(c) We can gauge fix the theory by introducing a gauge-fixing fermion Ψ. The gauge fixed action may be written as

$$S = \int d^4x\,\mathcal{L} + \int d^4x\,Q(\Psi),$$

where $\mathcal{L}$ is the Lagrangian given in $(\star)$, Q is the BRST charge and $Q(\Psi)$ denotes the BRST transformation of Ψ.

(i) Write down BRST transformations for the fields A_μ and ϕ.

(ii) Explain why the action is BRST invariant.

(iii) Show that the choice of gauge does not affect the correlation functions of gauge-invariant operators.

END OF PAPER