UNIVERSITY OF CAMBRIDGE

MAMA/303, NST3AS/303, MAAS/303

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 1:30 pm to 3:30 pm

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) A microscopic model consists of N spins on a lattice where each spin S_i can take values -1, 0 or $+1$. The energy is

$$E = -J \sum_{\langle ij \rangle} S_i S_j + g \sum_i S_i^2 - B \sum_i S_i$$

with $J > 0$ and $g > 0$, where the first term is a sum over nearest neighbours.

(i) Let $m = \frac{1}{N} \sum_i \langle S_i \rangle$. Show that $m = \frac{1}{N\beta} \frac{\partial \log Z}{\partial B}$.

(ii) Write $S_i = m + (S_i - m)$ and neglect terms of order $(S_i - m)^2$ in the first sum above (but treat the other sums exactly) to show that the mean-field theory approximation to the partition function is

$$Z = e^{-\beta N J q m^2/2} \left[c_1 + c_2 \kappa \cosh(\beta(Jqm + B)) \right]^N$$

where $\kappa = e^{-\beta g}$, q is the number of nearest neighbours of each site and c_1, c_2 are constants that you should determine.

(iii) Use the relation in (i) to obtain an equation for m. Hence show that mean field theory predicts a phase transition at $B = 0$, $T = T_c$ and give an equation for T_c (you need not solve this equation).

[When explaining why you reject the trivial solution for $T < T_c$ you need only consider small m.]

(b) Coarse graining the Ising model (with $B = 0$) leads to a Landau-Ginzburg theory with effective free energy

$$F[m] = \int d^d x \left(\frac{\gamma(T)}{2} (\nabla m)^2 + \frac{1}{2} \alpha_2(T) m^2 + \frac{1}{4} \alpha_4(T) m^4 + \ldots \right)$$

(i) Explain briefly the principles that are used to determine the form of $F[m]$.

(ii) Use mean field theory to determine the critical exponents $\alpha, \beta, \gamma, \delta$ of this model.

[Recall $c \sim |T - T_c|^{-\alpha}$, $m \sim (T_c - T)^{\beta}$, $\chi \sim |T - T_c|^{-\gamma}$, $m \sim B^{1/\delta}$. For γ you need only consider $T > T_c$.]

(iii) Explain the Ginzburg criterion and its connection to the upper critical dimension.

[You may assume that, in the quadratic approximation, the correlation function of this model scales as $r^{-(d-2)}$ for $r < \xi$ and decays exponentially for $r > \xi$, where $\xi \sim \alpha_2^{-1/2}$ is the correlation length.]

2

(a) Consider a theory of a real scalar field ϕ with effective free energy of the form

$$F = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \dots \right)$$

where $\dots$ denotes interaction terms such as $g\phi^4$. Assume there is an ultraviolet cutoff Λ in momentum space.

(i) Describe the three steps of the renormalization group procedure. Why does this define a flow on the coupling constants of the theory?

(ii) How are beta functions defined? How are they used to define scaling dimensions of coupling constants at a fixed point?

(iii) Define what is meant for a coupling constant to be relevant, irrelevant or marginal. What is a critical surface and how does it explain the phenomenon of universality?

(b) Now assume that the interaction terms are $\lambda\phi^3 + g\phi^4$.

(i) Starting at $\mu^2 = \mu_0^2$, $\lambda = \lambda_0$ and $g = g_0$ with small λ_0 and g_0, perform the first step of the RG procedure to determine the contributions to μ'^2 to leading non-vanishing order in g_0 and λ_0.

[You may use Wick's theorem without proof and the fact that, when $\lambda_0 = g_0 = 0$, $\langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}') G_0(\mathbf{k})$ where $G_0(\mathbf{k}) = (\mathbf{k}^2 + \mu_0^2)^{-1}$. You may leave your final answer in integral form.]

(ii) What is the lowest order at which λ_0 appears as a correction to g? Draw a corresponding Feynman diagram. [You are not expected to compute the correction.]

(iii) If $\lambda_0 = 0$ then what is the lowest order at which g_0 appears as a correction to λ? [You are not expected to compute the correction.]

3

(a) Consider a critical point in the Ising universality class. Assume that, near the critical point, the most singular part of the free energy depends only on the correlation length. Derive a scaling law relating the critical exponents α and ν.

(b) Consider the $O(N)$ model in d dimensions:

$$F = \int d^d x \left(\frac{1}{2}(\nabla \boldsymbol{\phi})^2 + \frac{1}{2}\mu^2 \boldsymbol{\phi}^2 + g(\boldsymbol{\phi}^2)^2 + \dots \right)$$

where $\boldsymbol{\phi} = (\phi_1, \dots, \phi_N)$ with $N \geqslant 2$, and $\boldsymbol{\phi}^2 = \sum_a \phi_a \phi_a$, $(\nabla \boldsymbol{\phi})^2 = \sum_a \nabla \phi_a \cdot \nabla \phi_a$.

(i) Use this model to explain the meaning of the terms *spontaneous symmetry breaking* and *Goldstone boson*.

(ii) Use the quadratic approximation to show that Goldstone bosons have infinite correlation length.

(iii) What is the lower critical dimension for this model? Explain your answer.

(c) Consider a different theory with scalar fields (ϕ_1, ϕ_2, ϕ_3) and effective free energy

$$F = \int d^d x \left(\frac{1}{2} \sum_{i=1}^{3} (\nabla \phi_i)^2 + \frac{1}{2} \sum_{i=1}^{3} \mu_i^2 \phi_i^2 + \sum_{i,j=1}^{3} g_{ij} \phi_i^2 \phi_j^2 \right)$$

where g_{ij} is a real, symmetric, positive definite matrix.

(i) The theory is invariant under a $Z_2 \times O(2)$ symmetry where Z_2 acts as $\phi_1 \to -\phi_1$ and $O(2)$ rotates (ϕ_2, ϕ_3). What relations between the coupling constants does this symmetry imply? Explain your answer.

(ii) Assume that $g_{11} = g_{22} = g_{33} = g_{12} = g_{13} = g_{23} = g > 0$ and $\mu_3^2 = \mu_2^2 \neq \mu_1^2$. Use mean field theory to predict the phase diagram in the (μ_1^2, μ_2^2) plane. What are the unbroken symmetries in each phase? Which phases have a Goldstone boson?

END OF PAPER