UNIVERSITY OF CAMBRIDGE

MAMA/302, NST3AS/302, MAAS/302

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what it means for G to be a *Lie group*, clearly stating the group axioms in mathematical terms.

(b) Let $g(x)$, $g(y)$, and $g(z)$ be elements of G depending on coordinates x, y, and $z \in \mathbb{R}^n$. Suppose that these group elements are close to the identity in G, which corresponds to the origin in the coordinate patch used. Assuming $g(z) = g(y)g(x)$, discuss how z can be related to x and y and the implications of the group axioms.

(c) Consider the group commutator $g(w) := g(y)^{-1}g(x)^{-1}g(y)g(x)$. Relate w to x and y using a Taylor expansion to leading, nonvanishing order.

(d) Consider the set of matrices which have the following form,

$$S := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

With reference to the group axioms, determine the largest subset of S that forms a group, G, under matrix multiplication, specifying clearly any conditions on a and b that may be required.

(i) What is the underlying manifold of G? Is it connected or disconnected? Justify your answers.

(ii) Is the defining representation of the group given above reducible? If so, identify an invariant subspace.

(iii) Consider the following subgroups of G,

$$H_0 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a > 0 \right\}, \quad H_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a > 0 \right\}, \quad H_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}.$$

For each H_n above ($n = 0, 1, 2$), explain whether H_n is normal (justifying your answer) and, if it is, determine the quotient group G/H_n.

2

This question concerns the Poincaré group in Minkowski spacetime with 1 time and 2 space dimensions. Under this group, spacetime coordinates transform as

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu \,,$$

where $\mu, \nu \in \{0, 1, 2\}$, and a^μ and $\Lambda^\mu{}_\nu$ are parameters of such transformations. Take the Minkowski metric signature to be $(+, -, -)$.

(a) State the defining condition which a 3×3 matrix Λ must satisfy in order to be an element of the Lorentz group in $1 + 2$ dimensions. Show what this condition implies for the determinant of Λ and the temporal-temporal component of Λ.

(b) Recall that $SL(2, \mathbb{R})$ is the group of 2×2 real matrices with determinant 1. Show that there is a group homomorphism from $SL(2, \mathbb{R})$ to the proper, orthochronous Lorentz group in $1 + 2$ dimensions, finding an explicit expression for the map. Is this map an isomorphism?

[Hint: consider the space $\mathcal{M}$ of real 2×2 matrices spanned by $\{\, \tau_0, \tau_1, \tau_2 \,\}$, with

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Construct a map ϕ from Minkowski space in $1 + 2$ dimensions to $\mathcal{M}$ such that $\phi(x) = X$, where $\det X = x^2$.]

(c) Write the Poincaré group in $1 + 2$ dimensions as a semidirect product, and give a definition of the subgroups known as *little groups*. [A full derivation is not expected here, but comment on why the semidirect product nature of the group is important.]

(d) Determine the unitary irreducible representations of the Poincaré group suitable for describing states of a single *massive* particle.

(e) Determine the unitary irreducible representations of the Poincaré group suitable for describing states of a single *massless* particle.

3

This question concerns the Lie algebra $\mathfrak{su}(2)$ and its finite-dimensional, irreducible representations. You may find the following identities useful. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given an angle θ and a 3-component unit vector $\mathbf{n}$,

$$\exp\left(\frac{i\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}}{2}\right) = I\cos\frac{\theta}{2} + i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}.$$

(a) Give a definition for the matrix Lie group $SU(2)$ and derive conditions determining which 2×2 matrices are elements of the Lie algebra $\mathfrak{su}(2)$. Determine the structure constants for the basis $\{ T_1, T_2, T_3 \}$, where

$$T_k = -\frac{i}{2}\sigma_k \quad \text{for } k = 1, 2, 3.$$

(b) Using the Cartan–Weyl basis $H = 2iT_3$, $E_\pm = i(T_1 \pm iT_2)$, construct an eigenvector basis for the finite-dimensional, irreducible representation d_Λ whose highest weight is Λ. Hence deduce the dimension of this space in terms of Λ.

(c) Take as given the fact that for some angle θ and unit vector $\mathbf{n}$ the following *disentangling identity* holds:

$$\exp\left(\frac{i\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}}{2}\right) = e^{i\alpha E_-}\exp\left(\frac{i\beta H}{2}\right)e^{i\gamma E_+}$$

for some α, β, and γ which are generally complex. Taking $\mathbf{n} = (1,0,0)$, determine α, γ, and $\exp(i\beta H/2)$ in terms of θ. [Do not attempt to solve for β itself.]

(d) Returning to the irreducible representation d_Λ of part (b), let us denote the Λ-eigenvector of $d_\Lambda(H)$ by $|\Lambda\rangle$, i.e. $d_\Lambda(H)\,|\Lambda\rangle = \Lambda\,|\Lambda\rangle$. Determine $\left\langle \Lambda \,\middle|\, e^{-\theta d(T_1)} \,\middle|\, \Lambda \right\rangle$ as a function of θ.

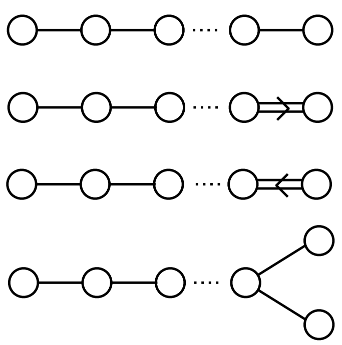
(e) For any angle θ define a corresponding state vector

$$|\theta\rangle := e^{-\theta d(T_1)}\,|\Lambda\rangle\,.$$

Determine the inner product between two such states, i.e. $\langle \varphi \,|\, \theta \rangle$. [Hint: Consider a disentangling identity similar to the one in part (c).]

4

This question concerns Cartan's classification of simple, complex, finite-dimensional Lie algebras. Consider the following families of algebras of rank r, each line giving their classification according to Cartan along with the corresponding Dynkin diagram and complexified Lie algebra:



A_r : $\mathfrak{su}(r+1)_{\mathbb{C}}$

B_r : $\mathfrak{so}(2r+1)_{\mathbb{C}}$

C_r : $\mathfrak{sp}(2r)_{\mathbb{C}}$

D_r : $\mathfrak{so}(2r)_{\mathbb{C}}$

The convention is that an arrow points from the longer root to the shorter.

In this question you may use any general results for simple, complex, finite-dimensional Lie algebras and their representations as long as they are clearly and correctly stated.

(a) Given that a rank r Lie algebra $\mathfrak{g}$ has simple roots $\{\, \alpha_{(j)} \,\}$ with $j = 1, 2, \ldots, r$, give definitions for the following terms: (i) *Cartan matrix*, (ii) *fundamental weights*, (iii) *Dynkin labels*, (iv) *dominant weight*, (v) *highest weight of a representation.* Additionally, (vi) determine the linear relation between the simple roots and the fundamental weights.

(b) Consider the rank-3 Lie algebras denoted by A_3, B_3, and C_3. In each case

(i) write down the corresponding Cartan matrix, assuming the labelling of roots is from left to right in its Dynkin diagram as depicted above;

(ii) give the ratio $|\alpha_{(2)}|/|\alpha_{(3)}|$;

(iii) give the angles between each pair of simple roots.

(c) Find all of the weights for the following representations and write down the dimension of each representation:

(i) d_1, the A_3 representation whose highest weight has Dynkin labels $[1, 0, 0]$;

(ii) d_2, the A_3 representation whose highest weight has Dynkin labels $[0, 0, 1]$;

(iii) d_3, the B_3 representation whose highest weight has Dynkin labels $[0, 0, 1]$.

(d) Decompose the A_3 representation $d_1 \otimes d_2$ into irreducible representations of A_3.

(e) Let us define a map P on Dynkin labels such that $P[\lambda^1, \lambda^2, \lambda^3] = [\lambda^2, \lambda^1, \lambda^2 + \lambda^3]$. Apply this to the weights of the representation d_3 found in (c)(iii) and note any similarities to the weights of d_1 and d_2 found in (c)(i) and (c)(ii). How can you interpret this?

END OF PAPER