MATHEMATICAL TRIPOS Part III

Thursday 5 June 2025 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider, in four dimensions, a massless Dirac spinor ψ coupled to a Maxwell gauge field A_μ via the following Lagrangian

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\bar{\psi}\,[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\,\psi\,.$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)$, γ^μ are the gamma matrices, and λ and e are real coupling constants.

(a) Are the coupling constants λ and e relevant, marginal or irrelevant? Explain your answer.

(b) State how $\psi(x)$, $\bar{\psi}(x)$, and $A_\mu(x)$ transform under a Lorentz transformation. Briefly explain why the Lagrangian is Lorentz invariant. Show that $\mathcal{L}$ is real up to total derivatives.

(c) Derive the equations of motion of the fields.

(d) Show that the theory is invariant under a local and global $U(1)$ symmetry. Derive the appropriate Noether current j^μ and charge Q. Show explicitly that Q is conserved on-shell. Compare j^μ with the source appearing in Maxwell's equation, and comment on any terms they differ by.

(e) Consider the transformation $\psi \to e^{i\alpha\gamma_5}\psi$ with α a constant and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Under what condition is this a symmetry of $\mathcal{L}$? Does the same condition apply if the fermion is massive? Is it possible to make this symmetry local by suitable choices of e and λ? Explain your answer.

2

(a) Consider a real vector field $V^\mu(x)$ in four-dimensional Minkowski space, with Lagrangian

$$\mathcal{L} = a\,(\partial_\mu V_\nu)(\partial^\mu V^\nu) + b\,(\partial_\mu V_\nu)(\partial^\nu V^\mu) + c\,(\partial_\mu V^\mu)(\partial_\nu V^\nu) + m^2 V^\mu V_\mu \ ,$$

where a, b, c, and m are real constants.

(i) Show that $\mathcal{L}$ can be simplified, up to total derivatives, to

$$\mathcal{L} = \tilde{a}\, V_\nu \partial_\mu \partial^\mu V^\nu + \tilde{b}\, V^\nu \partial_\nu \partial_\mu V^\mu + m^2 V^\mu V_\mu \ , \qquad (\star)$$

with $\tilde{a}$, $\tilde{b}$ real constants.

(ii) Using $(\star)$, derive the equations of motion for V^μ.

(iii) Using Noether's theorem, construct the energy-momentum tensor associated to this Lagrangian, and give an expression for the Hamiltonian.

(iv) Consider the case with $\tilde{b} = 0$. Show that for any value of $\tilde{a}$, the energy of the system is not positive definite.

(v) Assuming $\tilde{a} > 0$, find a relationship between $\tilde{b}$ and $\tilde{a}$ such that the energy of the system is positive definite on-shell, up to total derivatives. Show that for this choice the equations of motion of the vector field imply that $\partial_\mu V^\mu = 0$.

(b) Another way to constrain $\tilde{a}$ and $\tilde{b}$ to their physical values, is as follows. A general vector field can always be decomposed as

$$V_\mu = A_\mu + \partial_\mu \pi \ ,$$

where $A_\mu(x)$ satisfies $\partial_\mu A^\mu = 0$ (transverse condition), and $\pi(x)$ is a scalar field.

(i) Write the Lagrangian $(\star)$ in terms of A_μ and π. Derive the equations of motion for A_μ and π.

(ii) Show that the propagator of π, up to a numerical factor, is

$$\langle 0|\mathrm{T}\pi(x)\pi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{m^2}\left(\frac{1}{k^2} - \frac{\tilde{c}}{\tilde{c}k^2 - m^2}\right) e^{-ik\cdot(x-y)} \ ,$$

where $\tilde{c}$ is a constant that you should specify. [*Hint: You may use without proof the fact that the propagator is a Green's function.*]

Show that the term involving $\tilde{c}$ vanishes if $\tilde{a}$ and $\tilde{b}$ are related as in part (a)(v) of this question.

3 In this question we will consider a four-dimensional theory consisting of a massive scalar field $\phi(x)$ and a massive Dirac spinor field $\psi(x)$, and the effects of two different interactions between them. The free part of the Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi \ .$$

Here M and m are the masses of $\phi(x)$ and $\psi(x)$, respectively. We are using the notation $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and γ^μ are the gamma matrices.

(a) Consider the interacting theory with Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$, with

$$\mathcal{L}_1 = -\lambda\phi\bar{\psi}\psi \ ,$$

with λ a coupling constant.

(i) Write down the classical equations of motion. State, without derivation, the three Schwinger-Dyson equations for this theory.

(ii) Following the LSZ approach, derive the S-matrix for the decay $\phi \to \psi\bar{\psi}$ by defining asymptotic states and identifying the appropriate correlation function that enters in the S-matrix.

(iii) To leading order in the coupling, evaluate the connected contribution to the correlation function appearing in $\phi \to \psi\bar{\psi}$ by using the Schwinger-Dyson equations. With this, evaluate the S-matrix element for this process to leading order, and identify the corresponding amplitude $\mathcal{A}_{s,s'}$.

(iv) Compute the spin-summed/averaged squared matrix element,

$$\mathcal{P} := \frac{1}{4}\sum_{s,s'=1}^{2} |\mathcal{A}_{s,s'}|^2 \ ,$$

where s and s' are the spins of the final particles.

(b) Now consider instead the interacting theory with Lagrangian $\mathcal{L}_0 + \mathcal{L}_2$, with

$$\mathcal{L}_2 = -ig\phi\bar{\psi}\gamma^5\psi \ ,$$

with g a coupling constant and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

What is the S-matrix for $\phi \to \psi\bar{\psi}$? You do not need to derive the result, just state the appropriate modification of your answer in part (a)(iii) of this question.

Evaluate the spin-summed/averaged squared matrix element in this case. In an experiment, would you be able to detect the difference between the effect of the two interaction terms $\mathcal{L}_1$ and $\mathcal{L}_2$?

Useful identities: You may use without proof the following identities related to the mode expansion of free fields:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\tilde{\omega}_{\vec{p}}}}\left[a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x}\right] \ ,$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\sum_{r=1}^{2}\left[b^r_{\vec{p}}\, u^r(\vec{p})e^{-ip\cdot x} + c^{r\dagger}_{\vec{p}}\, v^r(\vec{p})\, e^{ip\cdot x}\right] \ ,$$

with $\tilde{\omega}_{\vec{p}}^2 = M^2 + \vec{p}^2$ and $\omega_{\vec{p}}^2 = m^2 + \vec{p}^2$. Some identities for spinors are

$$\begin{aligned}
(\not{p} - m)u^s(\vec{p}) &= 0\,, \\
(\not{p} + m)v^s(\vec{p}) &= 0\,,
\end{aligned}$$

$$\begin{aligned}
\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) &= \not{p} + m\,, \\
\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) &= \not{p} - m\,.
\end{aligned}$$

$$\begin{aligned}
u^{s\dagger}(\vec{p})u^{s'}(\vec{p}) &= v^{s\dagger}(\vec{p})v^{s'}(\vec{p}) &&= 2\omega_{\vec{p}}\,\delta^{ss'}\,, \\
\bar{u}^{s}(\vec{p})u^{s'}(\vec{p}) &= -\bar{v}^{s}(\vec{p})v^{s'}(\vec{p}) &&= 2m\,\delta^{ss'}\,, \\
u^{s\dagger}(\vec{p})v^{s'}(-\vec{p}) &= v^{s\dagger}(\vec{p})u^{s'}(-\vec{p}) &&= 0\,.
\end{aligned}$$

4 Suppose that we have a massive scalar field in four dimensions with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{2}\Phi^2 \ .$$

Consider the field redefinition

$$\Phi(x) = \varphi(x) + \lambda\, \varphi(x)^2 \ ,$$

where λ is a real constant. The scattering amplitudes for particles created by Φ or φ should be unchanged if we make this field redefinition. To check that a field redefinition does not change physical observables, do the following:

(a) Work out the Lagrangian in terms of the redefined field φ. Identify the free and interactions terms in the Lagrangian.

(b) For the Lagrangian in terms of φ, write, without derivation, the corresponding Feynman rules in momentum space for the S-matrix.

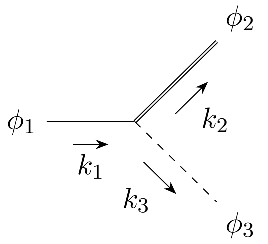
(c) Consider the scattering process $\varphi\varphi \to \varphi\varphi$. Draw the connected Feynman diagrams that contribute to the S-matrix to leading order in λ.

(d) Based on your answers to part (c) and (b), evaluate the connected contribution to $\varphi\varphi \to \varphi\varphi$ scattering at leading order in λ. Comment on your final answer.

[*Hint:* In part (b), you may use the fact that if three scalar fields interact via

$$\mathcal{L}_{\text{int}} = \lambda \phi_1 (\partial_\mu \phi_2)(\partial^\mu \phi_3) \ ,$$

then the rule for the vertex is



$$= (i\lambda)(ik_2)_\mu (ik_3)^\mu (2\pi)^4 \delta^4(k_1 - k_2 - k_3) \ .$$

Consider carefully how this can be applied when the three scalars are identical.]

END OF PAPER