UNIVERSITY OF CAMBRIDGE

MAMA/215, NST3AS/215, MAAS/215

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 1:30 pm to 4:30 pm

PAPER 215

MIXING TIMES OF MARKOV CHAINS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a Markov chain X on a finite state space S with invariant distribution π. Under which conditions is a randomised stopping time τ corresponding to this chain also a *strong stationary time*?

(b) We consider the following way in which k people shuffle a deck of n cards. One shuffle consists of each person taking a card which is currently on the top of the deck until $n-k$ cards are left in the deck, and then, one by one, in random order, each of the k people places their card in the uniformly random location in the deck (to return their card the first person chooses from $n-k+1$ locations, the second person chooses from $n-k+2$, and so on until the final person chooses from n locations). Let S be the set of all permutations of $\{1,\ldots,n\}$, X the Markov chain on S corresponding to this method of shuffling and π its invariant distribution.

(i) Check that π is the uniform distribution on S.

(ii) Let τ be a random time, which is the first time such that at time $\tau-1$, there are at most $k-1$ cards above the card which was initially the bottom card of the deck. Explain why τ is a strong stationary time.

(iii) For $k \ll n$, show that for all $\varepsilon \in (0,1)$ the mixing time of X, satisfies $t_{\text{mix}}(\varepsilon) \leqslant \frac{n \log n}{k} + Cn$ for large enough constant C depending on ε.

[You can use without proof any results from lectures provided that they are clearly stated. You can use without proof that the mean of a geometric random variable with a parameter q is $\frac{1}{q}$ while the variance is at most $\frac{1}{q^2}$ and that $\sum_{i=1}^{n} \frac{1}{i} = \log n + O(1)$.

Hint: In part (b)(iii) you may wish to consider some independent Bernoulli trials with success probabilities which differ for different trials and which are all at least p and couple suitably the time it takes to see the first success with a geometric random variable with parameter p.]

2

(a) Let X be a Markov chain on a finite state space S with invariant distribution π. For every $A \subset S$ set $\Phi(A) = \frac{\sum_{x \in A, y \in A^c} \pi(x)P(x,y)}{\pi(A)}$.

(i) Define the bottleneck ratio Φ_* and the isoperimetric profile $\Phi_*(r)$ for $r \leqslant \frac{1}{2}$.

(ii) Let $A = A_1 \cup A_2 \cup \ldots \cup A_k$ for some disjoint sets $A_i \subset S$ such that for all $i \neq j$ with $i, j \leqslant k$ for all $x \in A_i, y \in A_j$ we have that $P(x,y) = 0$. Show that $\Phi(A) \geqslant \inf_{i \leqslant k} \Phi(A_i)$.

(b) Let $V_1 = ([0, n^2] \times [0, n^2]) \cap \mathbb{Z}^2$ be the set of points with integer coordinates in a square box of side lengths n^2 and let E_1 be the lattice edges connecting (x_1, x_2) and (y_1, y_2) from V_1 if and only if $|x_1 - y_1| + |x_2 - y_2| = 1$. Let (V_2, E_2) be another copy of the pair (V_1, E_1). Let $E_{1,2}$ be a set of edges connecting for all $k, \ell \in \{0, 1, \ldots n\}$ the vertex with coordinates $(kn, \ell n)$, from V_1, with the vertex which has exactly the same coordinates in V_2 (there are $(n+1)^2$ edges in $E_{1,2}$). Let $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E_{1,2})$. Show that the mixing time of lazy, simple random walk on G satisfies $t_{\text{mix}} \lesssim n^4$.

[You are allowed to use any results from lectures without proof. For part (b), it might be useful to recall that for a lazy chain $t_{\text{mix}} \lesssim \int_{4\pi_{\min}}^{4/M} \frac{du}{u\Phi_*^2(u)} + \frac{1}{\gamma}\log(M)$ where γ is the spectral gap.]

3

(a) Suppose that a Markov chain with transition matrix P takes values in a finite vertex set V of a graph G with length function ℓ. Let ρ be the corresponding path metric, $\alpha \in \mathbb{R}$ and let

$$\rho_K(\mu, \nu) = \inf\{\mathbb{E}[\rho(X, Y)] : (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\}$$

be the transportation metric corresponding to ρ. Suppose that for all edges (x, y) there exists a coupling (X_1, Y_1) of $P(x, \cdot)$ and $P(y, \cdot)$ such that

$$\mathbb{E}_{x,y}[\rho(X_1, Y_1)] \leqslant e^{-\alpha}\rho(x, y).$$

Show that for any $x, y \in V$ we have $\rho_K(P(x, \cdot), P(y, \cdot)) \leqslant e^{-\alpha}\rho(x, y)$ and use this to show that for any probability measures μ and ν we have $\rho_K(\mu P, \nu P) \leqslant e^{-\alpha}\rho_K(\mu, \nu)$.

Let $\operatorname{diam}(V) = \max_{x,y} \rho(x, y)$ and deduce that

$$t_{\text{mix}}(\varepsilon) \leqslant \frac{1}{\alpha}(\log(\operatorname{diam}(V)) + \log(1/\varepsilon)).$$

[You can assume that a transportation metric corresponding to a path metric is a well-defined metric and that it upper bounds the total variation distance.]

(b) Let X be a Markov chain on state space of all subsets of $\{1, \ldots, n\}$ of size at most k, i.e. $V = \{A \subset \{1, 2, \ldots, n\} : |A| \leqslant k\}$, where $k < n$, which evolves as follows. If the current state is a set $A \in V$ we first pick m to be a uniform number from $\{0, 1\}$ and move the chain depending on the value of m. If $m = 0$ the chain stays in place. If $m = 1$ we pick a uniform number i in $\{1, \ldots, n\}$. If $i \in A$ the chain moves to $A\backslash\{i\}$, while if $i \notin A$, it moves to $A \cup \{i\}$ provided that $|A| < k$, and otherwise, it stays in place. For $A, B \in V$ let $\rho(A, B) = \frac{|A\backslash B|+|B\backslash A|+||A|-|B||}{2}$. Show that ρ is a path metric corresponding to some length function ℓ. Use it to show that the mixing time of X satisfies $t_{\text{mix}}(\varepsilon) \lesssim n \log\left(\frac{k}{\varepsilon}\right)$.

4

In this question, you can use any results from lectures without proof.

(a) For a finite, reversible, Markov chain X on the state space V define the quantity $\mathrm{hit}_\alpha(\varepsilon)$ for $\alpha, \varepsilon \in (0,1)$. Define a hit_α cutoff. Does mixing time cutoff imply a $\mathrm{hit}_{\frac{1}{2}}$ cutoff?

(b) Let $d > 0$ be an integer constant which does not depend on n and let $G = G_n = (V, E)$ be a graph for which $|V| = n$ is an even number and where each vertex has degree d. Let X be a simple random walk on G, let its mixing time be t_{mix} and let its relaxation time t_{rel} satisfy $t_{\mathrm{rel}} \ll t_{\mathrm{mix}}$. Let $\theta = \frac{1}{t_{\mathrm{mix}}}$ and let $\widetilde{E}$ be a set of edges on V, disjoint from E such that each vertex from V belongs to exactly one edge in $\widetilde{E}$. Let Y be a weighted random walk on $(V, E \cup \widetilde{E})$ such that edges in E have weight 1 and edges in $\widetilde{E}$ have weight θ, or more precisely Y is a Markov chain with transition probabilities $\widetilde{P}(x,y) = \mathbb{1}_{\{\{x,y\}\in E\}} \frac{1}{\theta+d} + \mathbb{1}_{\{\{x,y\}\in \widetilde{E}\}} \frac{\theta}{\theta+d}$ for all $x, y \in V$.

(i) Are X and Y reversible chains? Find their invariant distributions.

(ii) Prove that if X does not exhibit cutoff, then Y also does not exhibit cutoff.

[You can use without proof that for any constant c and any $f(n) \asymp \frac{1}{g(n)}$ such that $f(n) \to 0$ as $n \to \infty$ we have $(1 - cf(n))^{g(n)} \asymp 1$.]

5

(a) State Thomson's principle and define an edge-cutset. Prove the Nash-Williams inequality stating that if $(\Pi_k)_{k=1}^m$ is a sequence of m disjoint edge-cutsets which separate a and z then

$$R_{\mathrm{eff}}(a,z) \geqslant \sum_{k=1}^{m} \left(\sum_{e \in \Pi_k} c(e) \right)^{-1} .$$

State the commute time identity.

(b) Let $B = ([0,n] \times [0,n^2]) \cap \mathbb{Z}^2$ be the set of points with integer coordinates in a box of side lengths n and n^2 and let G be the graph with vertex set B and edges between (x_1, x_2) and (y_1, y_2) if and only if $|x_1 - y_1| + |x_2 - y_2| = 1$. Let X be a simple random walk on G starting from $(0,0)$ and let $T_{(n,n^2)}$ be the first time X visits the vertex (n, n^2). Find the order of $\mathbb{E}_{(0,0)}\left[T_{(n,n^2)}\right]$.

END OF PAPER