UNIVERSITY OF CAMBRIDGE

MAMA/209, NST3AS/209, MAAS/209

MAT3
MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 9:00 am to 12:00 pm

PAPER 209

RANDOM STRUCTURES IN FINITE-DIMENSIONAL SPACE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FIVE** questions in total.
The questions carry comparable weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Percolation in half-spaces

We consider site percolation with parameter p on the half-space graph $H = \mathbb{N} \times \mathbb{Z}^{d-1}$ for $d \geqslant 2$. The boundary hyperplane $\{0\} \times \mathbb{Z}^{d-1}$ will be denoted by B. So, any point in $H \setminus B$ has $2d$ neighbours in H, while points in B have $2d - 1$ neighbours in H.

Throughout this exercise, p is some fixed number in $(0, 1)$. The process $\omega = (\omega(x), x \in H)$ will be a collection of independent Bernoulli random variables of parameter p in H (so, this is site percolation in H with parameter p). We call $N = N(\omega)$ the number of infinite clusters for ω.

The goal of questions a) and b) is to derive some properties inspired by the proofs presented in the lectures for percolation in $\mathbb{Z}^d$:

a) Show that when p is large enough, there almost surely exists at least one open infinite cluster in H.

b) What is the law of the configuration $(\omega(x + e), x \in H)$ where $e = (0, 1, 0, \ldots, 0)$? Show that there exists $k_0 \in \mathbb{N} \cup \{\infty\}$ such that $N = k_0$ almost surely. Show that this k_0 is in fact in $\{0, 1, \infty\}$.

The goal of the next questions is to derive features that are more specific to the half-space case. We assume from now on that p is chosen in such a way that there exists (at least) one infinite open cluster for percolation in H.

c1) We consider percolation in H. We let A the event that the cluster C containing the origin is infinite, and that the origin is the only point in $B \cap C$ [equivalently, A is the event that an infinite cluster intersects B only at the origin]. We let $Q = P(A)$. Show that if $Q \neq 0$, then the probability that for percolation in the whole space $\mathbb{Z}^d$, there exists at least one infinite cluster contained in H and another infinite cluster contained in $H' := (-\infty, -2] \times \mathbb{Z}^{d-1}$ is at least $(1 - p)Q^2$. What can you conclude about Q *[You can use here the uniqueness of the infinite cluster for percolation in the whole-space proved in the lectures]*?

c2) Let $x_1, \ldots, x_n$ be n points in B for some $n \geqslant 2$. Can you adapt the proof of the previous result to show that for each infinite cluster C for percolation in H, the probability that $C \cap B = \{x_1, \ldots, x_n\}$ is equal to 0? Conclude that almost surely, any infinite cluster C for percolation in H that intersects B does so at infinitely many points.

d1) For each $n_0 \geqslant 0$, we define the half-space $H_{n_0} := [n_0, \infty) \times \mathbb{Z}^{d-1}$ and its boundary $B_{n_0} := \{n_0\} \times \mathbb{Z}^{d-1}$. We consider percolation in H_{n_0} that we then restrict to H_{n_0+1}. Suppose that C is an infinite cluster for the percolation restricted to H_{n_0+1} that intersects B_{n_0+1}. Using the result of the previous question, show that it is then necessarily contained in an infinite cluster C' for the percolation in H_{n_0} that intersects B_{n_0}.

d2) Suppose that with positive probability, there exists an infinite cluster C for percolation in H that does not intersect B. Using the previous question, show that this leads to a contradiction.

So, c2) and d2) show that any infinite cluster for percolation in H does intersect B at infinitely many points.

2 Large but finite percolation clusters

We consider supercritical site percolation $(w(x), x \in T)$ of parameter p for the triangular lattice T in the plane. The goal of this exercise is to estimate the probability that the origin is in a finite cluster that contains a large number of points.

Throughout this exercise, $p > 1/2$ is fixed.

The questions a1) to a4) deal with lower bounds:

We define $\theta = \theta(p)$ to be the probability that the origin is in an infinite open cluster.

When $N \geqslant 2$, we denote by H_N the big hexagon centered at the origin consisting of all the points on the triangular lattice that are at graph-distance at most N from the origin. We let $h_N = H_N \setminus H_{N-1}$.

We let C_N denote the set of points x in H_N such that there exists an open path from x to h_N.

We denote the number of points in a set S by $\#S$.

a1) Show that $E[\#C_N] \geqslant \theta \times \#H_N$.

a2) Deduce that for all $N \geqslant 2$, $P[\#C_N \geqslant (\theta/2) \times \#H_N] \geqslant \theta/2$ [*you may first look for an upper bound for* $E[\#C_N]$, *observing that* $\#C_N \leqslant \#H_N$ *always holds*].

a3) Let A_N denote the event $\{\#C_N \geqslant (\theta/2) \times \#H_N \text{ and } 0 \in C_N\}$. Show that for all $N \geqslant 2$, $P[A_N] \geqslant \theta^2/2$.

a4) Deduce that there exists a positive constant $u = u(p)$ such that the probability that the origin is in a finite cluster with at least $(\theta/2) \times \#H_N$ points is bounded from below by some constant times $\exp(-uN)$ *[you may consider the event where* A_N *holds and where all sites of* h_{N+1} *are open and where all the sites of* h_{N+2} *are closed].*

The following questions deal with upper bounds:

b1) Let $w'(x) = 1 - w(x)$. What can you say about $(w'(x), x \in T)$?

b2) Let B_n denote the event that there exists a circuit of closed sites for w with diameter at least n that surrounds the origin and goes through one of the n points $(1, 0), \ldots, (n, 0)$. Use exponential decay for subcritical percolation to show that for some $u(p) > 0$, $P[B_n] \leqslant n \exp(-un)$ for all $n \geqslant 1$.

b3) Use exponential decay for subcritical percolation to derive an exponential upper bound for the probability that there exists a closed circuit around the origin that goes through one of the points of $\{(m, 0),\ m \geqslant n\}$.

b4) Deduce an exponential upper bound for the probability that the origin is in a finite open cluster of diameter greater than n.

b5) Conclude that there exists $v(p) > 0$, such that for all $m \geqslant 1$, the probability that the origin is in a finite cluster with at least m sites is bounded by $\exp(-v\sqrt{m})$. *[you may use without proof the simple fact that for some absolute constant c, the diameter of a set with m points in the triangular lattice is at least* $c\sqrt{m}$*].* How does this compare with the result of a4)?

3 Harris-FKG inequality for the random cluster model

a) Write down the definition of the random cluster model (sometimes called FK-percolation) with parameter p associated to the Ising model on a finite graph, and detail how the coupling with an Ising model of parameter β (that depends on p) on this graph works.

b) Show that this random cluster probability measure is the stationary measure of a simple resampling Markov chain. Can you use this to compare (in some sense to be made precise) this random cluster measure with Bernoulli percolation with parameter p on the edges of the graph? Can you also compare this random cluster measure with Bernoulli percolation with parameter $p/(2-p)$ on the edges of the graph?

c) Let A and B be two non-empty increasing events (measurable with respect to the state of the edges of the graph). Using the previous Markov chain (and also a variant of this Markov chain), show that the random cluster measure satisfies the FKG-Harris inequality, namely that $P[A \cap B] \geqslant P[A]P[B]$. What can you say about $P[A \cap B']$ when A is an increasing event and B' a decreasing event?

d) Let P_N denote the random cluster measure on the finite box $\Lambda_N = \{-N, -N+1, \ldots, N-1, N\}^d$ (sometimes denoted as $[-N, N]^d$), viewed as part of the lattice $\mathbb{Z}^d$ (so that two points in Λ_N are neighbours if and only if they are neighbours in $\mathbb{Z}^d$). Let A denote an increasing non-empty event that is measurable with respect to the state of the edges in $[-N_0, N_0]^d$. Show that $(P_N[A], N \geqslant N_0)$ is non-decreasing with respect to N, and has therefore a limit as $N \to \infty$.

e) Can you compare this limit with the probability of A for Bernoulli percolation of parameter p on the edges of $\mathbb{Z}^d$ and with the probability of A for Bernoulli percolation of parameter $p/(2-p)$?

4 Coursework on the Green's function and Wilson's algorithm

Consider a finite connected graph D with $n+1$ sites $\{x_0, \ldots, x_n\}$. We assume that each point of D has the same number Δ of neighbours.

When $A \subset D$ is non-empty and $x \in D \setminus A$, we define $g_{D\setminus A}(x)$ to be the expected number of visits of x by a random walk in D started from x before it hits A.

a) Outline one possible proof of the fact that

$$\prod_{j=1}^{n} g_{D\setminus\{x_0,\ldots,x_{j-1}\}}(x_j)$$

does not depend on the chosen order of the points in D.

b) How does this quantity relate to the number of spanning trees in D. Explain in detail.

5 About conformal restriction and the continuum GFF

The two parts are independent.

PART A (ON THE CONTINUUM GAUSSIAN FREE FIELD (GFF))

A1) Recall the definition of the continuum Gaussian Free Field Γ in $\mathbb{R}^3$.

A2) Can one define the mean value of this GFF on the (two-dimensional) square $[0,1]^2 \times \{0\}$ (i.e., can one define the random variable $\Gamma(\mu)$ when μ is the uniform measure on this square)? Justify.

Can one define the mean value of this GFF on a (one-dimensional) segment $[0,1] \times \{0\} \times \{0\}$ (i.e., can one define the random variable $\Gamma(\mu)$ when μ is the uniform measure on this segment)? Justify.

PART B (ON CONFORMAL RESTRICTION).

We assume that there exists a measure ρ on the set of self-avoiding loops in the entire plane (viewed as a subset of the set of compact sets in the plane, endowed with the Hausdorff topology), such that the following holds:

- The ρ-mass of the set of loops of diameter in $[1/2, 1]$ that are contained in $[0,1]^2$ is equal to 1.
- For any two bounded simply connected domains D and D', and for any conformal transformation Φ from D onto D', the conformal image of ρ_D under Φ is equal to $\rho_{D'}$ (where ρ_D and $\rho_{D'}$ respectively denote the measure ρ restricted to the set of loops that are in D and the measure ρ restricted to the set of loops that stay in D').

B1) We denote by μ the measure ρ restricted to the set of loops that do surround the origin. What can you say about μ? *[You are allowed to use results discussed in the lectures without recalling their proofs].*

B2) Show that the measure ρ is invariant under translations of any vector x (that is under the mapping that associates to a loop γ the loop $\gamma + x$). Can you relate the measure μ_x defined as the measure ρ restricted to the set of loops that surround some given point x with the measure μ?

B3) Deduce that (provided it exists), the measure ρ is actually unique.

B4) Let x and y be two different points in $\mathbb{R}^2$ and D a bounded simply connected domain containing these two points. Show that the measure ρ restricted to the set of loops in D that surround both points x and y can be equivalently be described in terms of outer boundaries of Brownian loops that go through x or of outer boundaries of Brownian loops that go through y.

END OF PAPER