UNIVERSITY OF CAMBRIDGE

MAMA/202, NST3AS/202, MAAS/202, MGM3/30

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 9:00 am to 12:00 pm

PAPER 202

STOCHASTIC CALCULUS WITH APPLICATIONS TO FINANCE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For a càdlàg function $f : \mathbb{R}_+ \to \mathbb{R}$ of finite variation, define the following notation:

$$V_f(t) = \lim_n \sum_{k=1}^{\infty} |f(t \wedge t_k^n) - f(t \wedge t_{k-1}^n)| \text{ for } t \geqslant 0$$

where $t_k^n = k2^{-n}$,

$$\Delta_f(t) = f(t) - f(t-) \text{ for } t > 0,$$

where $f(t-) = \lim_{\varepsilon \downarrow 0} f(t-\varepsilon)$, and

$$A_f = \{t > 0 : \Delta_f(t) \neq 0\}\,.$$

(a) Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a distribution function. Show that F is càdlàg and that $V_F(t) = F(t) - F(0)$ for all $t \geqslant 0$. Show that the set A_F is countable. [Hint: you may wish to consider the sets $A^{(n)} = \{t \in [0,n] : \Delta_F(t) \geqslant 2^{-n}\}$.] Show that

$$F(t)^2 = F(0)^2 + 2\int_0^t F\, dF - \sum_{s \in A_F \cap (0,t]} |\Delta_f(s)|^2 \text{ for all } t \geqslant 0$$

where $\int_0^t F\, dF = \int_{\mathbb{R}_+} \mathbf{1}_{(0,t]} F\, dF$ as in lectures.

(b) Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ is càdlàg and of finite variation. Show that

$$V_f(t) - V_f(s) \geqslant |f(t) - f(s)| \text{ for all } 0 \leqslant s \leqslant t.$$

Hence show that there exist distribution functions F and G such that $f = F - G$. Conclude that the set A_f is countable and that

$$\sum_{s \in A_f \cap (0,t]} |\Delta_f(s)| < \infty \text{ for all } t \geqslant 0.$$

[You may use without proof the fact that V_f is a distribution function.]

(c) Suppose f and g are càdlàg and of finite variation. Show that

$$f(t)g(t) = f(0)g(0) + \int_0^t f\, dg + \int_0^t g\, df - \sum_{s \in A_f \cap A_g \cap (0,t]} \Delta_f(s)\Delta_g(s) \text{ for all } t \geqslant 0.$$

2 Let $(X_t)_{t\geqslant 0}$ be a continuous local martingale with $X_0 = 0$. For $t \geqslant 0$, let

$$A_t^{(n)} = \sum_{k=1}^{\infty} (X_{t\wedge t_k^n} - X_{t\wedge t_{k-1}^n})^2$$

and

$$M_t^{(n)} = X_t^2 - A_t^{(n)} = \sum_{k=1}^{\infty} 2X_{t_{k-1}^n}(X_{t\wedge t_k^n} - X_{t\wedge t_{k-1}^n})$$

where $t_k^n = k2^{-n}$. Let $\mathcal{M}^2$ denote the space of square-integrable continuous martingales with norm

$$\|M\| = \sup_{t\geqslant 0} \sqrt{\mathbb{E}(M_t^2)}.$$

(a) Under the assumption that there exists a constant $C > 0$ such that $|X_t(\omega)| \leqslant C$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$, show that

$$\text{(i)} \sup_{t\geqslant 0} \mathbb{E}(A_t^{(n)}) \leqslant C^2, \quad \text{(ii)} \sup_{t\geqslant 0} \mathbb{E}\big((M_t^{(n)})^2\big) \leqslant 4C^4, \quad \text{(iii)} \sup_{t\geqslant 0} \mathbb{E}\big((A_t^{(n)})^2\big) \leqslant 10C^4$$

Show that there exists a continuous adapted process A such that

$$\mathbb{E}\left(\sup_{t\geqslant 0}(A_t^{(n)} - A_t)^2\right) \to 0 \text{ as } n \to \infty.$$

Furthermore, show that if $A_t = 0$ almost surely for all $t \geqslant 0$ then $X_t = 0$ almost surely for all $t \geqslant 0$. [You may use without proof the fact that the sequence $(M^{(n)})_n$ is Cauchy in $\mathcal{M}^2$. You may also use the completeness of $\mathcal{M}^2$.]

(b) Show, without the assumption of uniform boundedness of X, that there exists a continuous adapted process A such that

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} |A_s^{(n)} - A_s| > \varepsilon\right) \to 0$$

for all $t \geqslant 0$ and $\varepsilon > 0$.

(c) Fix an exponent $0 < p < 2$ and for $t \geqslant 0$, let

$$B_t^{(n)} = \sum_{k=1}^{\infty} |X_{t\wedge t_k^n} - X_{t\wedge t_{k-1}^n}|^p.$$

Suppose that $\sup_n B_t^{(n)} < \infty$ almost surely for all $t \geqslant 0$. Show that $\sup_n B_t^{(n)} = 0$ almost surely for all $t \geqslant 0$.

3 (a) State Itô's formula for a d-dimensional continuous semimartingale X and twice-continuously differentiable $f : \mathbb{R}^d \to \mathbb{R}$.

(b) State and prove Lévy's characterisation of d-dimensional Brownian motion.

Let W be a scalar Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let

$$S_t = t^a + W_t$$

for an exponent $a > 0$. Fix non-random $T > 0$.

(c) Suppose $a > 1/2$. Show that there exists an equivalent probability measure $\mathbb{Q}$ such that the process $(S_t)_{0 \leqslant t \leqslant T}$ is a $\mathbb{Q}$-Brownian motion. [You may use the Cameron–Martin–Girsanov theorem and Novikov's criterion without proof.]

(d) Suppose $0 < a < 1/2$.

(i) Show that there exists a stopping time $\tau > 0$ such $S_t > 0$ for all $0 < t \leqslant \tau$. [You may use without proof the fact that $\liminf_{t \downarrow 0} t^{-a} W_t = 0$ almost surely.]

(ii) Show that there does *not* exist an equivalent probability measure $\mathbb{Q}$ such that the process $(S_t)_{0 \leqslant t \leqslant T}$ is a $\mathbb{Q}$-local martingale. [If you use a version of the fundamental theorem of asset pricing, you must prove it.]

4 Consider the stochastic differential equation

$$dX = b(X)dt + \sigma(X)dW \qquad (*)$$

where the measurable functions $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are given and W is scalar Brownian motion.

(a) What is a *strong solution* of equation $(*)$? A *weak solution*?

(b) What does it mean to say equation $(*)$ has the *pathwise uniqueness* property? The *uniqueness in law* property?

(c) Suppose that there is a constant $K > 0$ such that

$$2(x-y)(b(x)-b(y)) + (\sigma(x)-\sigma(y))^2 \leqslant K(x-y)^2$$

for all $x, y \in \mathbb{R}$. Show that equation $(*)$ has the pathwise uniqueness property. [You may use Gronwall's inequality in the form $f(t) \leqslant \alpha + \beta \int_0^t f(s)ds$ for $t \geqslant 0$ implies $f(t) \leqslant \alpha e^{\beta t}$ for $t \geqslant 0$.]

(d) Consider a weak solution of equation $(*)$ and let $U : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a bounded, twice-continuously differentiable solution to the partial differential equation

$$\frac{\partial U}{\partial t}(t,x) = b(x)\frac{\partial U}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2 \ \frac{\partial^2 U}{\partial x^2}(t,x)$$

(i) Show that

$$U(t, X_0) = \mathbb{E}(U(0, X_t)|X_0).$$

[If you use the Feynman–Kac formula, you must prove it.]

(ii) For fixed $t > 0$, show that there exists a previsible process $(\theta_s)_{0 \leqslant s \leqslant t}$ such that

$$U(0, X_t) = U(t, X_0) + \int_0^t \theta_s(dX_s - b(X_s)ds).$$

(e) Again consider a weak solution of equation $(*)$ and let $V : [\ell, r] \to \mathbb{R}$ be a bounded, twice-continuously differentiable solution to the ordinary differential equation

$$\lambda V(x) = b(x)V'(x) + \frac{1}{2}\sigma(x)^2 \ V''(x) \text{ for all } \ell \leqslant x \leqslant r,$$
$$V(\ell) = V(r) = 1$$

and where $\lambda > 0$ is a given constant. Let

$$T = \inf\{t \geqslant 0 : X_t < \ell \text{ or } X_t > r\}$$

Show on the event $\{\ell < X_0 < r\}$ that

$$V(X_0) = \mathbb{E}[e^{-\lambda T}|X_0]$$

where the notation $e^{-\lambda T} = 0$ on the event $\{T = \infty\}$ is used.

END OF PAPER