UNIVERSITY OF CAMBRIDGE

MAMA/201, NST3AS/201, MAAS/201

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 9:00 am to 12:00 pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a standard Brownian motion B in $\mathbf{R}$.

(b) Let B be a standard Brownian motion in $\mathbf{R}$.

(i) Fix $0 \leqslant a < b \leqslant 1$. Prove that $\int_a^b B_s ds$ is a Gaussian random variable with mean zero and variance $\int_a^b \int_a^b \min(s,t) ds dt$. *[Hint: approximate the integral using Riemann sums.]*

(ii) Suppose that $f \in L^2([0,1])$. Prove that $(B, f) := \int_0^1 B_s f(s) ds$ is a Gaussian random variable with mean zero and variance $\int_0^1 \int_0^1 f(r) f(s) \min(r,s) dr ds$. *[You may use without proof that the step functions are dense in $L^2([0,1])$.]*

(c) Suppose that (f_n) is an orthonormal basis of $L^2([0,1])$ and let (α_n) be a sequence of i.i.d. $N(0,1)$ random variables. For each $t \in [0,1]$ and $n \in \mathbb{N}$, let $X_n(t) = \int_0^t \sum_{i=1}^n \alpha_i f_i(s) ds$.

(i) For each $t \in [0,1]$, by first identifying the law of $X_n(t)$ or otherwise, show that $(X_n(t))_{n \in \mathbb{N}}$ converges in distribution as $n \to \infty$ to a $N(0,t)$ random variable.

(ii) For each $t \in [0,1]$, show that $(X_n(t))_{n \in \mathbb{N}}$ converges a.s. and in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ as $n \to \infty$ to a limit $X(t)$.

(iii) For each $0 \leqslant t_1 < \cdots < t_j \leqslant 1$, show that $(X(t_1), \ldots, X(t_j))$ has the same distribution as $(B_{t_1}, \ldots, B_{t_j})$ where B is a standard Brownian motion on $\mathbf{R}$.

2

(a) Let B be a standard Brownian motion in $\mathbf{R}$.

(i) State what it means for B to satisfy the strong Markov property.

(ii) Suppose that $T > 0$ is deterministic. Show that the process $(B_{T-t} - B_T)_{t\in[0,T]}$ has the same law as the process $(B_t)_{t\in[0,T]}$.

(iii) For each $T > 0$, let $I_T = \inf_{0\leqslant s\leqslant T} B_s$. Show that $\mathbf{P}[B_T = I_T] = 0$ for all $T > 0$ deterministic. *[You may use results proved in class provided you state them clearly.]*

(iv) Show that $B|_{[0,1]}$ a.s. attains its infimum exactly once.

(v) Deduce that a.s. the set of $t \in [0,1]$ at which $B|_{[0,1]}$ attains a local minimum is countable.

(b) Let B be a standard Brownian motion in $\mathbf{R}^2$ and for each $x \in \mathbf{R}^2$ let $\mathbf{P}_x$ be the law under which $B_0 = x$.

(i) For each $r \geqslant 0$ let $\tau_r = \inf\{t \geqslant 0 : |B_t| = r\}$. For $0 < \epsilon < |x| < R < \infty$, show that

$$\mathbf{P}_x[\tau_\epsilon < \tau_R] = \frac{\log R - \log|x|}{\log R - \log \epsilon}.$$

(ii) Show that the Lebesgue measure of $B([0,1])$ is a.s. equal to 0.

3 Let $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ be σ-algebras and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

(a) Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbf{P})$. Give the definition of $\mathbf{E}[X \,|\, \mathcal{G}]$.

(b) Show that if $X \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ then

$$\mathbf{E}[(X - \mathbf{E}[X \,|\, \mathcal{G}])^2] + \mathbf{E}[(\mathbf{E}[X \,|\, \mathcal{G}] - \mathbf{E}[X \,|\, \mathcal{H}])^2] = \mathbf{E}[(X - \mathbf{E}[X \,|\, \mathcal{H}])^2].$$

(c) By expanding $\mathbf{E}[(Y - \mathbf{E}[Y \,|\, \mathcal{G}])^2]$, show that if $Y \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathbf{E}[Y \,|\, \mathcal{G}] \stackrel{d}{=} Y$ then $\mathbf{E}[Y \,|\, \mathcal{G}] = Y$ a.s.

(d) State and prove the conditional version of Jensen's inequality.

(e) Show that if $Y \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathbf{E}[Y \,|\, \mathcal{G}] \stackrel{d}{=} Y$ then $\mathbf{E}[Y \,|\, \mathcal{G}] = Y$ a.s. *[You may assume without proof that there exists a strictly convex, differentiable function φ with $|\varphi(x)| \leqslant |x|$ for all x. Hint: apply the identity $\varphi(x) > \varphi'(m)(x - m) + \varphi(m)$ for all $m \neq x$ with $x = Y$ and $m = \mathbf{E}[Y \,|\, \mathcal{G}]$ to compare $\mathbf{E}[\varphi(Y) \,|\, \mathcal{G}]$ and $\varphi(\mathbf{E}[Y \,|\, \mathcal{G}])$ on $\{\mathbf{E}[Y \,|\, \mathcal{G}] \neq Y\}$.]*

4

(a) (i) State the a.s. martingale convergence theorem.

(ii) Let (X_n) be a non-negative supermartingale. Show that (X_n) converges a.s. to an a.s. finite limit.

(iii) Give an example of a martingale (X_n) with $X_n \to -\infty$ a.s.

(b) Let (X_n) be a martingale with $|X_{n+1} - X_n| \leqslant C$. Let $A = \{\lim_n X_n \text{ exists and is finite}\}$ and $B = \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}$. Show that $\mathbf{P}[A \cup B] = 1$. *[Hint: apply part* (aii) *to* $(X_{n\wedge\tau} + M)$ *for a carefully chosen collection of stopping times* τ *and constants* M*.]*

(c) Give an example of a martingale (X_n) with $|X_{n+1} - X_n| \leqslant C$ so that $\mathbf{P}[A], \mathbf{P}[B] > 0$ where A, B are as in the previous part.

(d) Events A_1, A_2 are said to be equal a.s. if $\mathbf{P}[A_1 \setminus A_2] = 0$ and $\mathbf{P}[A_2 \setminus A_1] = 0$. Let $(\mathcal{F}_n)$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let B_n be a sequence of events with $B_n \in \mathcal{F}_n$. Using part (b) or otherwise, show that the events

$$\{B_n \text{ occurs infinitely often}\} \quad \text{and} \quad \left\{ \sum_{n=1}^{\infty} \mathbf{P}[B_n \mid \mathcal{F}_{n-1}] = \infty \right\}$$

are equal a.s.

5

(a) Give the definition of what it means for a sequence of random variables (X_n) to converge in distribution to a random variable X.

(b) Suppose that (X_n), (Y_n) are independent sequences of random variables with values in $\mathbf{R}^d$ and $X_n \to X$ in distribution and $Y_n \to Y$ in distribution. Show that $X_n + Y_n \to X + Y$ in distribution.

(c) Suppose that (X_n) are i.i.d. real-valued random variables with characteristic function ϕ. For each $n \in \mathbb{N}$ let $S_n = X_1 + \cdots + X_n$. Show that if ϕ is differentiable at 0 with $\phi'(0) = ia$ then $S_n/n \to a$ in probability.

(d) Give the definition of what it means for family of probability measures (μ_n) on a metric space M to be tight.

(e) Suppose that $(X_t^n)_{n \geqslant 1}$ is a family of stochastic processes with values in the space $C([0,1])$ of real-valued continuous functions equipped with the metric $d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|$ such that for every $p > 0$ there exist constants $c, \epsilon > 0$ so that

$$\mathbf{E}[|X_t^n - X_s^n|^p] \leqslant c|t-s|^{1+\epsilon} \quad \text{for all} \quad s,t \in [0,1] \quad \text{and} \quad n \in \mathbb{N}$$

and $X_0^n = 0$ for all $n \in \mathbb{N}$. Let μ_n be the law of (X_t^n) on $C([0,1])$. Show that the family (μ_n) is tight. *[You may use without proof that for every $\alpha \in (0,1)$ and $C > 0$ the set $\mathcal{X}_{C,\alpha}$ of $f \in C([0,1])$ such that $f(0) = 0$ and $|f(s) - f(t)| \leqslant C|s-t|^\alpha$ for all $s,t \in [0,1]$ is compact in $C([0,1])$.]*

6

(a) Consider the space $C([0,1])$ of real-valued continuous functions equipped with the metric $d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$. Let $\mathcal{F}$ be the smallest σ-algebra on $C([0,1])$ which makes the maps $\pi_t \colon C([0,1]) \to \mathbf{R}$ defined by $\pi_t(f) = f(t)$ measurable for all $0 \leqslant t \leqslant 1$. Show that $\mathcal{F} = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra on $C([0,1])$. *[You may use without proof that $C([0,1])$ is separable.]*

(b) Let A be a closed set and let X be a continuous, adapted process. Show that $T_A = \inf\{t \geqslant 0 : X_t \in A\}$ is a stopping time.

(c) Suppose that (X_n) is a UI sequence with $X_n \to X$ a.s.

(i) Show that $\mathbf{E}[X_n \,|\, \mathcal{G}] \to \mathbf{E}[X \,|\, \mathcal{G}]$ in L^1.

(ii) Let (Y_n), (Z_n) be independent sequences of independent random variables with

$$\mathbf{P}[Y_n = 1] = \frac{1}{n}, \quad \mathbf{P}[Y_n = 0] = 1 - \frac{1}{n}, \quad \mathbf{P}[Z_n = n] = \frac{1}{n}, \quad \mathbf{P}[Z_n = 0] = 1 - \frac{1}{n}.$$

By considering $X_n = Y_n Z_n$ or otherwise, show that it need not be true that $\mathbf{E}[X_n \,|\, \mathcal{G}] \to \mathbf{E}[X \,|\, \mathcal{G}]$ a.s.

(d) (i) State the Skorokhod embedding theorem.

(ii) Suppose that S_n is a square integrable martingale with $S_0 = 0$. Suppose that B a Brownian motion. Show that there exist stopping times $0 = T_0 \leqslant T_1 \leqslant T_2 \leqslant \cdots$ for B so that

$$(S_0, \ldots, S_k) \stackrel{d}{=} (B(T_0), \ldots, B(T_k)) \quad \text{for all} \quad k \geqslant 0.$$

END OF PAPER