UNIVERSITY OF CAMBRIDGE

MAMA/167, NST3AS/167, MAAS/167

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 9:00 am to 12:00 pm

PAPER 167

INTRODUCTION TO GEOMETRIC REPRESENTATION THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

All algebraic varieties are defined over an algebraically closed field k of characteristic p, $p \geqslant 0$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Define what it means for G to be an affine algebraic group.

(ii) Prove that any such has a faithful finite dimensional representation V.

(iii) For each of a) $G = \mathbb{G}_a$, and b) $G = \mathbb{G}_m$,

Decompose $k[G]$ as a representation of $G \times G$, where $G \times G$ acts by left and right multiplication on G.

Let V be a finite dimensional representation of G.

What is $Hom_G(k[G], V)$? What is $Hom_G(V, k[G])$?

2

Let G be an affine algebraic group, defined (as always in this exam) over an algebraically closed field k. Let $g \in G(k)$.

a) Define what it means for g to be (i) *semisimple*, and (ii) *unipotent*. Define the *Jordan decomposition* g.

Give an example of an algebraic group G with non-trivial semisimple and unipotent elements such that the semisimple elements are dense in G.

In your example, do they form an open subvariety?

Give an example of such a group where the semisimple elements are not dense in G, but form a closed subvariety.

[You must justify your answers!]

b) Define the derived subgroup $[G, G]$ of G, and show that it is a connected algebraic group if G is connected.

Suppose G is solvable. Show every element of $[G, G]$ is unipotent, and that $[G, G]$ is nilpotent as a group.

Define diagonalisable, unipotent, semisimple and reductive algebraic groups.

c) Prove Kolchin's theorem, that unipotent algebraic groups are nilpotent as a group.

Give an example to show that nilpotent groups aren't necessarily unipotent.

3 i) Let H be a closed subgroup of an affine algebraic group G, and I the ideal of functions vanishing on H. Construct a finite dimensional subspace $W \subset I$ whose stabiliser is H, and hence prove there exists a finite dimensional representation V of G and line $\ell \in \mathbf{P}V$ whose stabiliser is H.

ii) Define what it means for a map $X \to Y$ to be a flat H-torsor. Briefly sketch why the orbit map $G \to G\ell$, $g \mapsto g\ell$, is a flat H-torsor, stating clearly the results of lectures you are using.

iii) If $G = SL_2$ and B is a Borel, prove that the quotient map $G \to G/B$ is a Zariski B-torsor.

iv) If $G = \mathbb{G}_m$ and $H = \mu_2$, is the quotient map a Zariski μ_2-torsor? Describe explicitly a representation V of G and $\ell \in \mathbf{P}V$ as in (i).

4

Let $V = k^{2n}$, equipped with the symplectic form given by $\langle e_i, e_{2n-i}\rangle = 1$ if $i \leqslant n$, and $\langle e_i, e_j\rangle = 0$ if $i + j \neq 2n$.

Let $G = Sp(V) = Sp_{2n}$ be the group of automorphisms preserving the form.

Let $\mathcal{B} = \{F_1 < F_2 < \cdots < F_n \mid \dim F_i = i, F_i \leqslant F_i^{\perp}\}$.

(i) Show that $\mathcal{B}$ is a projective algebraic variety, and that G acts on it transitively. Show that the stabiliser of a point $F \in \mathcal{B}$ is a maximal closed connected solvable subgroup of G.

Pick such a subgroup $B = stab_G(F)$, and choose a maximal torus $T \subseteq B$.

(ii) Let $\mathfrak{g} = Lie(G)$ be the Lie algebra of G. Write the root space decomposition of $\mathfrak{g}$, and write the root datum $\Phi \subseteq X^*(T), \Phi^\vee \subseteq X_*(T)$.

(iii) Describe the Weyl group $W = N(T)/T$.

(iv) State and prove the Bruhat decomposition for Sp_{2n}, explicitly.

For any subset $I = \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$, let $\mathcal{P}_I = \{F_{i_1} < \cdots < F_{i_r} \mid \dim F_{i_j} = i_j, F_{i_j} \leqslant F_{i_j}^{\perp}\}$, and $\pi_I : \mathcal{B} \to \mathcal{P}_I$ the natural projection.

(v) Describe $stab_G(\pi_I(F))$ for each I, and identify the fiber $\pi_I^{-1}(F_I)$, $F_I := \pi_I(F)$, as a flag variety of an explicit affine algebraic group.

(vi) Let Z_G be the center of G. Identify this as an affine algebraic group, and prove that the quotient group $PSp(V) = G/Z_G$ is an affine algebraic group.

Write its root datum.

[Throughout this question, you may freely quote any theorems of lectures or of basic algebraic geometry, provided that you state them explicitly.]

END OF PAPER