UNIVERSITY OF CAMBRIDGE

MAMA/166, NST3AS/166, MAAS/166

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 9:00 am to 12:00 pm

PAPER 166

DIOPHANTINE ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Questions 1 and 3 are each worth 30 marks.
Question 2 is worth 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State two theorems giving lower bounds for linear forms in logarithms of algebraic numbers. One of them should be general, the other should be an improvement in the special case when the form is homogeneous with integer coefficients.

(b) Prove that there is an effective absolute constant C, such that $|2^n - 3^m| \geqslant 2^n/n^C$ for all $n, m \in \mathbb{Z}_{\geqslant 2}$.

(c) The Fibonacci numbers are a sequence of integers $F_1, F_2, \ldots$ that is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$. Prove that there is an effective, absolute constant C such that there is no perfect k-th power among the Fibonacci numbers for any $k > C$. You may use without proof the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

2

(a) Define the *height* of an algebraic number. [This is also known as the absolute or Weil height.] In your answer, include an explanation of the places of a number field and the normalization of the corresponding absolute values.

(b) State and prove the *product formula* for the places of a number field.

(c) Let α be an algebraic number and let $k \in \mathbb{Z}$. State and prove an identity for $H(\alpha^k)$ in terms of $H(\alpha)$.

(d) Let $\alpha \neq 0$ be an algebraic number. State and prove upper and lower bounds for $|\alpha|$ in terms of $H(\alpha)$ and the degree of α. [This is also known as the Liouville bound.]

(e) Let $k \in \mathbb{Z}_{\geqslant 1}$, $n_1, \ldots, n_k \in \mathbb{Z}_{\geqslant 0}$ and let $P, Q \in \mathbb{Z}[X_1, \ldots, X_k]$ be two polynomials that are of degree at most n_j in the variable X_j for each j. Let $\alpha_1, \ldots, \alpha_k$ be algebraic numbers. State without proof an upper bound for

$$H\left(\frac{P(\alpha_1, \ldots, \alpha_k)}{Q(\alpha_1, \ldots, \alpha_k)}\right).$$

Deduce upper bounds for $H(\alpha+\beta)$ and $H(\alpha+1)$ in terms of $H(\alpha)$ and $H(\beta)$, where α and β are algebraic numbers.

(f) Compute the height of a rational number p/q if $\gcd(p, q) = 1$. [You may use without proof any result from the lectures if you state it precisely.]

(g) For each $C \in \mathbb{R}_{>0}$, find algebraic numbers α and β such that $H(\alpha + \beta) > C(H(\alpha) + H(\beta))$.

(h) Prove that there is $\delta \in \mathbb{R}_{>0}$ such that for all $C \in \mathbb{R}_{>0}$, there is an algebraic number α such that $H(\alpha + 1) > (1 + \delta)H(\alpha) + C$.

3 In this question, α is an algebraic number of degree $d \geqslant 3$.

(a) State Siegel's lemma including the definition of height of linear forms.

(b) Let $\kappa > 0$. Prove that there is a constant $C_1 = C_1(\alpha, \kappa) < \infty$ depending only on α and κ such that for all $n \in \mathbb{Z}_{>0}$, there are polynomials $P, Q \in \mathbb{Z}[X]$ of degree at most n, not both 0, such that $H(P), H(Q) \leqslant C_1^n$ and

$$\frac{d^k}{k! dX^k}(P(X) + \alpha Q(X))\Big|_{X=\alpha} = 0$$

for $0 \leqslant k < (2-\kappa)n/d - 1$.

(c) Prove that the polynomials P and Q constructed in Part (b) are linearly independent over $\mathbb{Q}$ provided $\kappa < 1$ and n is sufficiently large depending on κ and d.

(d) Let $F(X,Y) = P(X) + YQ(X) \in \mathbb{Z}[X,Y]$ be a polynomial that is of degree at most n in X, such that $H(F) \leqslant C^n$, where $n \in \mathbb{Z}_{>1}$ and C is a positive real number. Suppose that P and Q are linearly independent over $\mathbb{Q}$. Let $\delta > 0$. Prove that if $p/q \in \mathbb{Q}$ is a rational number with $\gcd(p,q) = 1$ and q is sufficiently large depending only on C and δ, then for any fixed $y \in \mathbb{R}$, $F(X,y)$ has a zero of multiplicity at most $\delta n + 1$ at $X = p/q$.

[*Hint: Consider the function*

$$\begin{vmatrix} F(X,Y) & \frac{\partial}{\partial Y}F(X,Y) \\ \frac{\partial}{\partial X}F(X,Y) & \frac{\partial^2}{\partial X \partial Y}F(X,Y) \end{vmatrix}.$$

You may use without proof any result about Wronskians from the lectures.]

(e) Fix some $\kappa > 0$. Let $p_1/q_1 \in \mathbb{Q}$ with $\gcd(p_1, q_1) = 1$ and let $y \in \mathbb{R}$. Let $n \in \mathbb{Z}_{>0}$. Prove that if q_1 and n are both sufficiently large depending on α, κ, then there is $C_2 = C_2(\alpha, \kappa)$ such that the following holds. There is a non-zero polynomial $F(X,Y) = P(X) - YQ(X) \in \mathbb{Z}[X,Y]$ of degree at most n in X with $H(F) \leqslant C_2^n$ such that $F(X,\alpha)$ vanishes to order at least $(2-\kappa)n/d$ at $X = \alpha$ and $F(p_1/q_1, y) \neq 0$.

(f) In this part, you may use the following fact without proof. Suppose that p_1/q_1, y, n, κ satisfy all the assumptions made in Part (e), and, therefore, we may find some F and C_2 that satisfy the conclusion of Part (e). Then there is a constant $C_3 = C_3(\alpha, \kappa, C_2)$ such that

$$|F(p_1/q_1, y)| < (|\alpha - p_1/q_1|^{(2-\kappa)n/d} + |\alpha - y|)C_3^n.$$

Show that given $\varepsilon > 0$, there are only finitely many rational numbers p/q such that $|\alpha - p/q| < q^{-d/2-1-\varepsilon}$.

[If you use any results about Diophantine approximation of algebraic numbers from the lectures, you need to give a full proof.]

END OF PAPER