UNIVERSITY OF CAMBRIDGE

MAMA/165, NST3AS/165, MAAS/165

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 9:00 am to 12:00 pm

PAPER 165

SIMPLICIAL HOMOTOPY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define the category Δ and the category of simplicial sets. For an integer $n \geqslant 0$, define the simplicial set Δ^n. If $0 \leqslant i \leqslant n$, define the simplicial set Λ^n_i.

(ii) Given a simplicial set X, what does it mean for X to be a quasicategory? What does it mean for X to be a Kan complex?

(iii) Given a category $\mathcal{C}$, there is an associated simplicial set $N(\mathcal{C})$ called the nerve of $\mathcal{C}$. For each integer $n \geqslant 0$, describe the set of maps of simplicial sets from Δ^n to $N(\mathcal{C})$ in terms of the objects and morphisms of $\mathcal{C}$.

(iv) Suppose that $\mathcal{C}$ is a category and Q is a quasicategory. Prove that any map of simplicial sets $p : Q \to N(\mathcal{C})$ admits lifts against the inner horn inclusion $\Lambda^3_1 \to \Delta^3$.

(v) Suppose that

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout square of simplicial sets, such that the map $A \to B$ is a monomorphism. For every Kan complex X, the induced diagram of simplicial sets

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(D, X) & \longrightarrow & \underline{\mathrm{Hom}}(B, X) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(C, X) & \longrightarrow & \underline{\mathrm{Hom}}(A, X) \end{array}$$

is a pullback square. Prove that it is also a homotopy pullback square.

2

(i) Define the normalized chains functor N_* from simplicial abelian groups to chain complexes.

(ii) Let K denote the Eilenberg–MacLane functor from chain complexes to simplicial abelian groups, which is right adjoint to the functor N_*. State precisely the Dold–Kan correspondence, in terms of the restriction of K to a subcategory of chain complexes.

(iii) Let C_* denote the chain complex

$$\cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0 \to 0 \to \cdots$$

In terms of the groups C_i and the maps ∂_i, give an explicit formula for the set of maps of simplicial sets from Λ^2_1 to $K(C_*)$.

(iv) If A is an abelian group and $n \geqslant 0$ an integer, write down the definition of the Kan complex $K(A, n)$ in terms of the functor K.

(v) Suppose

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is a short exact sequence of abelian groups. Prove that there is a homotopy fiber sequence

$$K(A_1, n) \to K(A_2, n) \to K(A_3, n)$$

of pointed Kan complexes, for any integer $n \geqslant 0$.

(vi) Suppose that $f : K(\mathbb{Z}/4, 3) \times K(\mathbb{Z}/5, 3) \to K(\mathbb{Z}/4, 3) \times K(\mathbb{Z}/5, 3)$ is a map of pointed Kan complexes, such that

$$H_3(f; \mathbb{Z}) : H_3(K(\mathbb{Z}/4, 3) \times K(\mathbb{Z}/5, 3); \mathbb{Z}) \to H_3(K(\mathbb{Z}/4, 3) \times K(\mathbb{Z}/5, 3); \mathbb{Z})$$

is surjective. Prove that f is a homotopy equivalence.

3

(i) Compute the cohomology groups $H^*(\Omega S^4; \mathbb{Q})$. You do not need to compute the cup product structure.

(ii) Prove that $H^i(K(\mathbb{Z}, 3); \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ when $i = 0$ or 3, but trivial for all other integers i. You may assume knowledge of the cohomology ring of $K(\mathbb{Z}, 2) \simeq \mathrm{Sing}(\mathbb{CP}^\infty)$.

(iii) What is the smallest value of $i > 0$ for which $H_i(\Omega S^4; \mathbb{Z})$ is a non-zero group? What is that group?

(iv) Compute $\pi_i(\Omega S^4) \otimes_{\mathbb{Z}} \mathbb{Q}$ for each integer $0 \leqslant i \leqslant 6$. [*If it helps you to translate between cohomological and homological information, you may assume that the homology groups of* ΩS^4 *and* $K(\mathbb{Z}, 3)$ *are finitely generated abelian groups.*]

(v) What is $\pi_7(S^4) \otimes_{\mathbb{Z}} \mathbb{Q}$?

4

The exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to \mathbb{Z}/4 \to 0$$

gives rise to a homotopy fiber sequence

$$K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2) \to K(\mathbb{Z}/4, 2)$$

of pointed Kan complexes. Extending this sequence once to the left gives another homotopy fiber sequence

$$K(\mathbb{Z}/4, 1) \xrightarrow{f} K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2).$$

Note that we use f to denote the first map in this fiber sequence.

(i) The above fiber sequence gives rise to a Serre spectral sequence beginning with

$$\mathrm{E}_2^{p,q} = H^p(K(\mathbb{Z}, 2); H^q(K(\mathbb{Z}/4, 1); \mathbb{F}_2))$$

Calculate the groups $\mathrm{E}_r^{p,q}$ for every integer $r \geqslant 2$ and every pair of nonnegative integers (p, q) such that $p + q \leqslant 2$. You may assume knowledge of the cohomology ring of $K(\mathbb{Z}, 2) \simeq \mathrm{Sing}(\mathbb{CP}^\infty)$.

As a consequence of your calculation, write down the cohomology groups $H^0(K(\mathbb{Z}/4, 1); \mathbb{F}_2)$, $H^1(K(\mathbb{Z}/4, 1); \mathbb{F}_2)$, and $H^2(K(\mathbb{Z}/4, 1); \mathbb{F}_2)$. In the last of these groups, what is the image of the map $H^2(f; \mathbb{F}_2)$?

(ii) Consider the map $Sq^1 : K(\mathbb{F}_2, 1) \to K(\mathbb{F}_2, 2)$ of pointed Kan complexes. Prove that the homotopy fiber of this map is equivalent to $K(\mathbb{Z}/4, 1)$. You may assume knowledge of the cohomology ring of $K(\mathbb{F}_2, 1) \simeq \mathrm{Sing}(\mathbb{RP}^\infty)$. [*One possible strategy is to first prove that the homotopy fiber must be equivalent to either* $K(\mathbb{F}_2, 1) \times K(\mathbb{F}_2, 1)$ *or* $K(\mathbb{Z}/4, 1)$, *which can then be distinguished using cohomology with* $\mathbb{F}_2$ *coefficients.*]

(iii) In the commutative ring $H^*(K(\mathbb{Z}/4, 1); \mathbb{F}_2)$, prove that degree 1 classes square to zero. In other words, if $x \in H^1(K(\mathbb{Z}/4, 1); \mathbb{F}_2)$, prove that $x^2 = 0$.

END OF PAPER