UNIVERSITY OF CAMBRIDGE

MAMA/164, NST3AS/164, MAAS/164

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 17 June 2025 9:00 am to 11:00 am

PAPER 164

ENTROPY METHODS IN COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Starting from the axioms for entropy, prove that entropy is subadditive: that is, $\mathbf{H}[X,Y] \leqslant \mathbf{H}[X] + \mathbf{H}[Y]$ for any two discrete random variables X and Y taking values in a finite set. Prove also the submodularity rule for entropy.

(ii) Show that for any three random variables X, Y and Z we have the inequality

$$\mathbf{H}[X,Y,Z] \leqslant \frac{1}{2}\Big(\mathbf{H}[X,Y] + \mathbf{H}[Y,Z] + \mathbf{H}[Z,X]\Big) - \frac{1}{6}\Big(\mathbf{I}[X:Y] + \mathbf{I}[Y:Z] + \mathbf{I}[Z:X]\Big),$$

where $\mathbf{I}$ stands for mutual information.

(iii) Proving any facts you might need along the way, establish the the submodularity rule for sums: that if X, Y and Z are independent discrete random variables taking values in a finite Abelian group G, then

$$\mathbf{H}[X+Y+Z] - \mathbf{H}[Y+Z] \leqslant \mathbf{H}[X+Z] - \mathbf{H}[Z].$$

(iv) Define Ruzsa distance, and prove that under the same conditions as in part (iii),

$$\mathbf{H}[X+Y-Z] - \mathbf{H}[X-Y] \leqslant d[X;Z] + d[Y;Z] - d[X;Y].$$

2 Let G be a bipartite graph with vertex sets X and Y and density α. Let T be a tree with vertex set V of size k, partitioned into sets V_0 and V_1 in such a way that every edge of T joins a vertex in V_0 to a vertex in V_1. Let ϕ be a random map from V to $X \cup Y$ such that $\phi(V_0) \subset X$ and $\phi(V_1) \subset Y$, chosen uniformly at random from all such maps. Say that ϕ is a *homomorphism* if $\phi(x)\phi(y)$ is an edge of G for every edge xy of T. Prove that the probability that ϕ is a homomorphism is at least α^{k-1}.

3 (i) Prove that if $\mathcal{A}$ is a union-closed family of subsets of $\{1, 2, \ldots, n\}$ that does not consist just of the empty set, then there is some x that is contained in at least $(3-\sqrt{5})|\mathcal{A}|/2$ of the sets in $\mathcal{A}$. [You may assume the inequality $h(xy) \geqslant \phi(xh(y) + yh(x))/2$, where h is the binary entropy function and $\phi = (1+\sqrt{5})/2$ is the golden ratio.]

(ii) Let $\mathcal{A}$ be a family of subsets of $\{1, 2, \ldots, n\}$ of size k. Show that the number of quintuples of sets $(A_1, A_2, A_3, A_4, A_5)$ such that all pairwise unions $A_i \cup A_j$ (with $i \neq j$) belong to $\mathcal{A}$ is at most $3^{5k/2}|\mathcal{A}|^{5/2}$. [The sets $A_1, \ldots, A_5$ are not assumed to belong to $\mathcal{A}$: they are arbitrary subsets of $\{1, 2, \ldots, n\}$.]

4 (i) State and prove the entropic Balog-Szemerédi-Gowers theorem.

(ii) Let X, Y, U and V be $\mathbb{F}_2^n$-valued random variables and suppose that $d[U;X] + d[V;Y] \leqslant Cd[X;Y]$. Let U_1 and U_2 be copies of U and let V_1 and V_2 be copies of V, with U_1, U_2, V_1 and V_2 all independent. Prove that

$$d[U_1 + U_2 \mid U_1 + U_2 + V_1 + V_2; X] + d[V_1 + V_2 \mid U_1 + U_2 + V_1 + V_2; Y] \leqslant (aC + b)d[X;Y]$$

for some pair of absolute constants a, b. [You may assume the Ruzsa triangle inequality and appropriate submodularity results, but any other lemmas you might need should be proved.]

END OF PAPER