UNIVERSITY OF CAMBRIDGE

MAMA/163, NST3AS/163, MAAS/163

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 1:30 pm to 4:30 pm

PAPER 163

FOURIER RESTRICTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.

Question 1 carries 30 marks.
Question 2 carries 15 marks.
Question 3 carries 25 marks.
Question 4 carries 30 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Use the notation $e(t) = e^{2\pi i t}$. Consider $2 \leqslant p < \infty$ and $k \in \mathbb{N}$, $k \geqslant 2$. Let $C_{p,k}(N)$ be the infimum of $C > 0$ such that

$$\int_{[0,1]^{k-1}} \Big| \sum_{n=1}^{N} b_n e((n^2, \ldots, n^k) \cdot x) \Big|^p dx \leqslant C \Big(\sum_{n=1}^{N} |b_n|^2 \Big)^{\frac{p}{2}}$$

for any $b_n \in \mathbb{C}$.

(a) State Khintchine's inequality.

(b) Show that there exists $\tilde{C} \in (0, \infty)$, permitted to depend on p and k, such that for all $N \geqslant 1$,

$$\tilde{C}(1 + N^{\frac{p}{2} - \frac{k(k+1)}{2} + 1}) \leqslant C_{p,k}(N).$$

(c) Show that for all $\epsilon > 0$, there exists $C_\epsilon \in (0, \infty)$ such that

$$C_{p,2}(N) \leqslant C_\epsilon N^\epsilon (1 + N^{\frac{p}{2} - 2}).$$

Hint: *You may use that for any $\epsilon > 0$, there exists $D_\epsilon \in (0, \infty)$ such that for any integer $1 \leqslant m \leqslant M$, the number of divisors of m is $\leqslant D_\epsilon M^\epsilon$.*

2 Use the notation $e(t) = e^{2\pi i t}$. Let $N, T \geqslant 1$ and $2 \leqslant p < \infty$. Define $D_p(N,T)$ to be the infimum of $C > 0$ satisfying

$$\int_{[0,1]_x^2 \times [0,T]_t} \Big| \sum_{\substack{n=(n_1,n_2)\in\mathbb{Z}^2 \\ |n|\leqslant N}} b_n e(x\cdot n + t(\sqrt{2}n_1^2 + n_2^2)) \Big|^p dxdt \leqslant C \Big(\sum_{\substack{n=(n_1,n_2)\in\mathbb{Z}^2 \\ |n|\leqslant N}} |b_n|^2 \Big)^{\frac{p}{2}}$$

for all $b_n \in \mathbb{C}$.

(a) Let

$$f(x,t) = \sum_{\substack{n=(n_1,n_2)\in\mathbb{Z}^2 \\ |n|\leqslant N}} e(x\cdot n + t(\sqrt{2}n_1^2 + n_2^2)).$$

Show that there are constants $a_0 > 0$ and $C > 0$ and a discrete set $\mathcal{T}$ of times $t \in (0,T)$ satisfying

(a) $|f(0,t)| \geqslant a_0 N^2$ for all $t \in \mathcal{T}$,

(b) $|t - t'| \geqslant 1$ if $t, t' \in \mathcal{T}$ and $t \neq t'$, and

(c) $|\mathcal{T}| \geqslant CT/N^2$.

Hint: *You may assume that there is a constant $c_0 > 0$ so that the following is true: for each $\delta \in (0,1)$ and $T \geqslant 0$, we have*

$$\#\{b \in \{0,\dots,T\} : dist(b\sqrt{2},\mathbb{Z}) < \delta\} \geqslant c_0\delta T.$$

(b) Let

$$g(x,t) = \sum_{\substack{n=(n_1,n_2)\in\mathbb{Z}^2 \\ |n|\leqslant N}} b_n e(x\cdot n + t(\sqrt{2}n_1^2 + n_2^2)).$$

Show that there is an absolute constant $C_0 \in (0,\infty)$ so that for any $\epsilon > 0$, there is $C_\epsilon \in (0,\infty)$ satisfying the following: for any $(x_0,t_0) \in \mathbb{R}^3$, we have

$$|g(x_0,t_0)| \leqslant C_0 N^4 \int_{\mathcal{B}} |g(x,t)| dxdt + C_\epsilon N^{-1000} \|g\|_{L^\infty(\mathbb{R}^3)}$$

in which $\mathcal{B} = \{(x,t) \in \mathbb{R}^3 : |x - x_0| < N^{\epsilon-1}, \quad |t - t_0| < N^{\epsilon-2}\}$.

(c) Show the following constructive interference lower bound: for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$C_\epsilon N^{-\epsilon} T N^{p-6} \leqslant D_p(N,T)$$

for all $N \geqslant 1$ and $T \geqslant 1$.

3 Let $\gamma(t) = (t, t^2, t^3)$. For sets $A \subset \mathbb{R}^d$, let χ_A denote the characteristic function of A.

(a) State the trilinear Kakeya inequality in $\mathbb{R}^3$, for tubes with orientations in S^2 taken within $1/30$ of the standard unit basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

(b) Prove a version of the trilinear Kakeya inequality in $\mathbb{R}^3$ with directions determined by γ. That is, for a constant $c > 0$ that you may choose as you wish, state an estimate that holds uniformly for any three finite families $\mathcal{T}_j$, $j = 1, 2, 3$, of $R^{1/2} \times R^{1/2} \times R$ tubes (say cylindrical segments) in directions from the sets

$$\Sigma_j = \{\omega \in S^2 : \mathrm{dist}_{S^2}(\omega, \gamma(\tfrac{j}{3})/|\gamma(\tfrac{j}{3})|) < c\}.$$

Hint: *You may use without proof that for* $x_1, x_2, x_3 \in \mathbb{R}$,

$$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

(c) Prove that for all $\epsilon > 0$, there exists $C_\epsilon \in (0, \infty)$ such that the following holds for any $r \geqslant 1$: for any finite collections $\mathcal{S}_j$ of $r \times r \times 1$ blocks S_j with unit normal within $1/10$ of $\mathbf{e}_j$ and any $c_{S_j} \geqslant 0$, we have

$$\| \prod_{j=1}^{3} (\sum_{S_j \in \mathcal{S}_j} c_{S_j} \chi_{S_j}(x))^{1/3} \|_{L^3(B_r)} \leqslant C_\epsilon r^\epsilon \prod_{j=1}^{3} (\sum_{S_j \in \mathcal{S}_j} c_{S_j})^{1/3}$$

for any r-ball $B_r \subset \mathbb{R}^3$.

Hint: *Consider the size of a 3-fold intersection* $S_1 \cap S_2 \cap S_3$ *of blocks coming from each family* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$.

4

(a) State the decoupling inequality for the parabola, with exponents $2 \leqslant p < \infty$ and using norms $L^p(\mathbb{R}^2)$.

For parts (b) and (c), we first define some notation. Let $\gamma(t) = (t, t^2, t^3)$ and consider the cubic moment curve $\mathcal{M}^3 = \{\gamma(t) : 0 \leqslant t \leqslant 1\}$. To formulate decoupling for $\mathcal{M}^3$, for each $R \in 8^{\mathbb{N}}$, we define the collections $\Theta(R)$ which partition $\mathcal{M}^3$.

- Let $\Theta(R)$ be the collection of approximate $R^{-\frac{1}{3}} \times R^{-\frac{2}{3}} \times R^{-1}$ blocks θ, defined by the convex hull of $\{\gamma(t) : t \in I_\theta\}$, where $I_\theta \subset [0,1]$ is a dyadic interval of length $R^{-\frac{1}{3}}$.
- Note that for each $S \in 8^{\mathbb{N}}$ with $S < R$ and each $\tau \in \Theta(S)$, $\theta \in \Theta(R)$, either $\theta \subset \tau$ or $\theta \cap \tau = \emptyset$.
- For each $\theta \in \Theta(R)$, let $f_\theta : \mathbb{R}^3 \to \mathbb{C}$ be Schwartz functions satisfying $\mathrm{supp}\widehat{f_\theta} \subset \theta$.

(b) Let $r \in 8^{\mathbb{N}}$ satisfy $1 \leqslant r \leqslant R$ and suppose $w_{r^{1/3}} : \mathbb{R}^3 \to [0,\infty)$ is a Schwartz function such that $\widehat{w}_{r^{1/3}}$ is supported in $B_{r^{-1/3}}(0)$. Prove that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} |\sum_{\theta \in \Theta(R)} f_\theta(x) w_{r^{1/3}}(x)|^2 dx \leqslant C \sum_{\tau \in \Theta(r)} \int_{\mathbb{R}^3} |\sum_{\substack{\theta \in \Theta(R) \\ \theta \subset \tau}} f_\theta(x) w_{r^{1/3}}(x)|^2 dx.$$

(c) Let $r \in 8^{\mathbb{N}}$ satisfy $1 \leqslant r \leqslant R$ and suppose $w_{r^{2/3}} : \mathbb{R}^3 \to [0,\infty)$ is a Schwartz function such that $\widehat{w}_{r^{2/3}}$ is supported in $B_{r^{-2/3}}(0)$. Prove that for each $\epsilon > 0$, there exists $C_\epsilon \in (0,\infty)$ such that

$$\int_{\mathbb{R}^3} |\sum_{\theta \in \Theta(R)} f_\theta(x) w_{r^{2/3}}(x)|^6 dx \leqslant C_\epsilon \Big(\sum_{\tau \in \Theta(r)} \Big(\int_{\mathbb{R}^3} |\sum_{\substack{\theta \in \Theta(R) \\ \theta \subset \tau}} f_\theta(x) w_{r^{2/3}}(x)|^6 dx \Big)^{2/6} \Big)^{6/2}.$$

Hint: *Freeze the x_3-variable in $\sum_{\theta \in \Theta(R)} f_\theta(x_1, x_2, x_3) w_{r^{2/3}}(x_1, x_2, x_3)$ and apply part (a) to each slice.*

END OF PAPER