UNIVERSITY OF CAMBRIDGE

MAMA/162, NST3AS/162, MAAS/162

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 1:30 pm to 4:30 pm

PAPER 162

INTERSECTION THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let K be a field and X be a scheme of finite type over $\mathrm{Spec}(K)$.

(i) Let W be a $k+1$ dimensional subvariety of X, $r \in R(W)^*$ a non-zero rational function on W and $V \subseteq W$ a k-dimensional subvariety.

Define the order $\mathrm{ord}_V(r)$ of r along V, the divisor $[\mathrm{div}(r)]$ of r in $Z_k(X)$, the notion of rationally equivalent k-cycles and finally the k-th Chow group $A_k(X)$ of X.

(ii) Let $j : Z \hookrightarrow X$ be a closed subscheme and $i : U \hookrightarrow X$ its complement. Prove that for all k there is an exact sequence

$$A_k(Z) \xrightarrow{j_*} A_k(X) \xrightarrow{i^*} A_k(U) \to 0$$

[You are allowed to assume that proper pushforwards and flat pullbacks are well-defined in Chow].

(iii) A *cellular decomposition* of X is a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset$$

by closed subschemes such that each difference $X_i \setminus X_{i-1}$ is a disjoint union of schemes $U_{i,j}$ isomorphic to affine spaces $\mathbb{A}^{n_{i,j}}$. Let $V_{i,j}$ denote the closure of $U_{i,j}$ in X.

State (without proof) the Chow ring of affine spaces $\mathbb{A}^n$, and show that $A_*(X)$ is generated by the classes $[V_{i,j}]$. Finally, use this to compute the Chow groups of $\mathbb{P}^r$.

(iv) Suppose that X is the blow-up of $\mathbb{P}^r$ at a point p. Let E denote the exceptional divisor and let $\pi : X \to \mathbb{P}^r$ be the blow-up map. For each $k = 0, \ldots, r-1$, let $\Gamma_k \subseteq \mathsf{E}$ be a linear subspace of dimension k, and for $j = 1, \ldots, r$, let H_j be a linear subspace of dimension j of $\mathbb{P}^r$ passing through p. Denote by $\widetilde{H}_j \subseteq X$ the strict transform of H_j (recall that, by definition, this is the closure of $\pi^{-1}(H_j) \setminus \mathsf{E}$ in X).

Show that the classes $[\Gamma_k]$ for $k = 0, \ldots, r-1$ and $[\widetilde{H}_j]$ for $j = 1, \ldots, r$ generate $A_*(X)$.

2 Let $\pi : E \to X$ be a rank $r = e+1$ vector bundle on a scheme X and denote by $p : \mathbb{P}(E) \to X$ the associated projective bundle.

(i) Define isomorphisms

$$\Theta_E : \bigoplus_{i=0}^{e} A_{k-e+i}(X) \to A_k(\mathbb{P}(E)).$$

[You are not required to prove that the map is an isomorphism; you only need to define it. Moreover, you do not need to define Chern classes.]

(ii) Using the fact that the above maps are isomorphisms, deduce that the flat pullback $\pi^* : A_*(X) \to A_{*+r+1}(E)$ is injective.

(iii) Let $0 \to \mathcal{O}(-1) \to p^*E \to \xi \to 0$ be the tautological exact sequence over $\mathbb{P}(E)$. Show that for every $\alpha \in A_*(X)$ we have

$$p_*(c_i(\xi) \cap p^*\alpha) = \begin{cases} \alpha \text{ if } i = e \\ 0 \text{ if } i < e. \end{cases}$$

(iv) Let $\zeta = c_1(\mathcal{O}(1))$. Show that for every $\beta \in A_k(\mathbb{P}(E))$ we have the identity

$$\{c(p^*E) \cap \sum_{j\geqslant 0} \zeta^j \cap p^*p_*(\sum_{i\geqslant 0} \zeta^i \cap \beta)\}_k = \beta.$$

3

(i) State the Euler sequence on $\mathbb{P}^m$ and use it to express the total Chern class

$$c(T_{\mathbb{P}^m}) = \sum_{i\geqslant 0} c_i(T_{\mathbb{P}^m})$$

as a polynomial in $\mathsf{H} = c_1(\mathcal{O}(1))$. [You do not need to prove that the Euler sequence is exact, nor define the maps involved; you only need to state the sequence.]

(ii) Let $X \subseteq \mathbb{P}^m$ be a pure k-dimensional closed subscheme of $\mathbb{P}^m$. Give two characterizations of the degree $\deg(X \subseteq \mathbb{P}^m)$ of X in $\mathbb{P}^m$: one using Hilbert polynomials and one as an integral on $\mathbb{P}^m$. [You do not need to prove that the two agree or any of the intermediate results necessary to formulate them, as this was done in class.] Explain why $\deg(X \subseteq \mathbb{P}^m)$ is always a positive integer.

(iii) Suppose that X is a smooth variety of dimension n and that T_X is trivial (i.e., $T_X \cong \mathcal{O}^n$). Show that if there exists a closed embedding $X \hookrightarrow \mathbb{P}^m$, then $m \geqslant 2n$.

(iv) Let X be as in (iii), and suppose that it admits a closed embedding $X \hookrightarrow \mathbb{P}^{2n}$. Show that its degree satisfies $\deg(X \subseteq \mathbb{P}^{2n}) = \binom{2n+1}{n}$.

END OF PAPER