UNIVERSITY OF CAMBRIDGE

MAMA/150, NST3AS/150, MAAS/150

MAT3
MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 9:00 am to 12:00 pm

PAPER 150

ANALYTIC NUMBER THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the sieving function $S(A, \mathbb{P}, z)$. Prove that if $A \subset \mathbb{Z}$ is finite and $z \geqslant 2$, then we have

$$S(A, \mathbb{P}, z) = \sum_{d|P(z)} \mu(d) |\{a \in A \colon a \equiv 0 \pmod{d}\}|,$$

where $P(z) = \prod_{p \leqslant z} p$.

(b) Show that for any $2 \leqslant z \leqslant \log x$ we have

$$|\{n \in [1, x] \colon n \text{ has no prime factors} \leqslant z\}| = (C + o(1)) \frac{x}{\log z}$$

for some constant $C > 0$.

(c) Show that for $x \geqslant 2$ we have

$$|\{n \in [1, x] \colon 2n + 1, 3n + 1, 5n + 1 \text{ are all primes}\}| \ll \frac{x}{(\log x)^3}.$$

[*In parts (b) and (c), you may use Mertens' theorem and any results about sieves stated in the course.*]

2

(a) State Perron's formula. Show that for any $x \geqslant 2$ with $x - 1/2$ an integer, and any $T \in [2, x^2]$, we have

$$\sum_{n \leqslant x} \Lambda(n) = -\frac{1}{2\pi i} \int_{1+1/\log x - iT}^{1+1/\log x + iT} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{x(\log x)^2}{T}\right).$$

(b) Show that there exists a constant $c > 0$ such that for any $x \geqslant 2$ we have

$$\sum_{n \leqslant x} \Lambda(n) = x + O(x \exp(-c\sqrt{\log x})).$$

(c) For $\mathrm{Re}(s) > 1$ and $u \in \mathbb{R}$, write the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) n^{iu}}{n^s}$$

in terms of the Riemann zeta function.

(d) Suppose that there exists $u \in \mathbb{R}$ such that for all $x \geqslant 2$ we have

$$\sum_{n \leqslant x} \Lambda(n) n^{iu} = \frac{x^{1+iu}}{1+iu} + O(x^{1/2}(\log x)^2).$$

Show that the Riemann hypothesis holds.

[*You may use any results stated in the course, except for the the prime number theorem, the explicit formula, or equivalences for the Riemann hypothesis.*]

3 Let $f\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function with $|f(n)| \leqslant 1$ for all $n \in \mathbb{N}$. Let

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

(a) Show that the series $D_f(s)$ converges and defines an analytic function in the region $\mathrm{Re}(s) > 1$.

(b) Define the pretentious distance between two arithmetic functions.

(c) Write the Euler product formula for $D_f(s)$. By taking logarithms of the Euler product, or otherwise, show that for $x \geqslant 2$ and $t \in \mathbb{R}$ we have

$$\left| D_f\left(1 + \frac{1}{\log x} + it\right)\right| \asymp (\log x)\exp(-\mathbb{D}(f, n^{it}; x)^2).$$

(d) Assume that the series $D_f(s)$ and $D_{f^2}(s)$ converge for $\mathrm{Re}(s) > 1-c$ for some constant $c > 0$ and define analytic functions in this region. By considering the expression

$$\zeta(\sigma)^3 |D_f(\sigma + it)|^4 |D_{f^2}(\sigma + 2it)|,$$

or otherwise, show that $D_f(s) \neq 0$ whenever $\mathrm{Re}(s) \geqslant 1$.

[*In part (a), you may use the fact that a uniform limit of analytic functions in analytic. In part (c), you may use any estimates for primes stated in the course. In part (d), you may use any results stated in the course.*]

END OF PAPER