UNIVERSITY OF CAMBRIDGE

MAMA/144, NST3AS/144, MAAS/144

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 9:00 am to 11:00 am

PAPER 144

MODEL THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let T be a complete theory in a countable language $\mathcal{L}$, and let κ be an infinite cardinal.

(a) Define what it means for T to be κ-categorical.

(b) Let $\mathcal{L} = \{+, 0\}$ be the language of abelian groups, $\mathbb{Z}$ the integers and B be the group of functions $f \colon \mathbb{N} \to \{0, 1\}$ such that $(f + g)(n) := f(n) + g(n) \pmod 2$ and $\mathbf{0}(n) := 0$ for all $n \in \mathbb{N}$. Determine whether the following theories are $\aleph_0$-categorical.

(i) $\mathrm{Th}(\mathbb{Z}, +, 0)$.

(ii) $\mathrm{Th}(B, +, \mathbf{0})$.

For (ii), you may use standard results about Abelian groups provided that you state them correctly and precisely.

Justify your answers.

2 Let T be a complete theory in a countable language $\mathcal{L}$, and let κ be an infinite cardinal.

(a) Define what it means for $\mathcal{U}$ to be:

(i) a filter on $\mathbb{N}$.

(ii) an ultrafilter on $\mathbb{N}$.

(b) Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Show that either:

(i) $\{C \subseteq \mathbb{N} : \mathbb{N} \setminus C \text{ is finite}\} \subseteq \mathcal{U}$, or

(ii) there is some $n \in \mathbb{N}$ such that $\mathcal{U} = \{A \subseteq \mathbb{N} : n \in A\}$.

(c) Let $\mathcal{C}_i$ be the $\mathcal{L}_{\text{groups}}$ structure $(\mathbb{Z}/i\mathbb{Z}, +, 0)$, where $+$ denotes addition modulo i. Let $\mathcal{U}$ be an ultrafilter and suppose $\mathcal{C} = \prod_{i \in \mathbb{N}} \mathcal{C}_i / \mathcal{U}$. Is it possible to choose $\mathcal{U}$ such that:

(i) $\mathcal{C}$ is finite?

(ii) $\mathcal{C}$ has a definable function $f : \mathcal{C} \to \mathcal{C}$ that is surjective, but not injective?

(iii) $\mathcal{C}$ has an element with infinite order?

Justify your answers.

3 Let $\mathcal{L}$ be the language of orders. Recall that the theory DLO is the theory of dense linear orders without endpoints. Let $\mathcal{Q} = (\mathbb{Q}, <)$ be the $\mathcal{L}$-structure where $<$ is interpreted as the usual ordering on $\mathbb{Q}$.

(a) Show that DLO has quantifier elimination (*you may assume equivalents of quantifier elimination from lectures if stated correctly and precisely*).

(b) Give the definition of

(i) the set $S_1^{\mathcal{Q}}(\mathbb{N})$.

(ii) an isolated type in $S_1^{\mathcal{Q}}(\mathbb{N})$.

(c) Consider $S_1^{\mathcal{Q}}(\mathbb{N})$.

(i) Give the formulas that isolate the isolated types in $S_1^{\mathcal{Q}}(\mathbb{N})$.

(ii) What are the non-isolated types in $S_1^{\mathcal{Q}}(\mathbb{N})$?

Justify your answers.

END OF PAPER