UNIVERSITY OF CAMBRIDGE

MAMA/136, NST3AS/136, MAAS/136

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 9:00 am to 12:00 pm

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let L/K be a finite extension of non-archimedean local fields. Show that L/K is totally ramified if and only if $L = K(\alpha)$ where α is a root of an Eisenstein polynomial.

(b) Let L/K be a finite Galois extension of local fields. Define the higher ramification groups $G_s(L/K)$ for $s \geqslant -1$. Show that if $M \subset L$ is a subextension of L/K, we have $G_s(L/M) = G_s(L/K) \cap \mathrm{Gal}(L/M)$. Show also that if M/K is Galois, the image of $G_0(L/K)$ in $\mathrm{Gal}(M/K)$ is $G_0(M/K)$.

Calculate the higher ramification groups for the case $K = \mathbb{Q}_3$ and L is the splitting field of the polynomial $f(X) = X^3 - 3$.

2

(a) Let K be a field and let $|.|_1$ and $|.|_2$ be non-trivial absolute values which induce the same topology on K. Show that there exists $c \in \mathbb{R}_{>0}$ such that $|.|_1 = |.|_2^c$.

(b) Let $(K, |.|)$ be a non-archimedean valued field. Define the valuation ring $\mathcal{O}_K$ and maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$, and show that if $\mathcal{O}_K$ is Noetherian, then the associated valuation v is discrete. Let $S \subset K$ be the set of elements x such that $1 + x$ has an n^{th} root in K for infinitely many n. Show that

(i) $S \subset \mathcal{O}_K$ if v is discrete.

(ii) $\mathfrak{m} \subset S$ if K is complete and discretely valued.

Now assume K is complete and discretely valued, and let v' be another discrete valuation on K. Show that $v(a) = 0$ implies $v'(a) \geqslant 0$. Deduce that v is equivalent to v'.

3 (a) State and prove a version of Hensel's Lemma and use it to describe the group of roots of unity in $\mathbb{Q}_p$.

(b) Let $p > 2$ be a prime. Show that the equation

$$x^p + y^p = z^p$$

has a solution with x, y, z in $\mathbb{Z}_p^\times$ if and only if there exists an integer a such that $p \nmid a(a+1)$ and we have

$$(a+1)^p \equiv a^p + 1 \mod p^2.$$

4

(a) Let $(K, |.|)$ be a valued field and L/K a finite extension. Let $|.|_L$ be an absolute value on L extending $|.|$ on K.

(i) Show that $|.|$ is non-archimedean if and only if $|.|_L$ is non-archimedean.

(ii) Suppose that K is complete and discretely valued. Show that any absolute value on L extending $|.|$ is equivalent to $|.|_L$.

(b) Let $L = \mathbb{Q}(\alpha)$ where α is a root of the polynomial $f(X) = X^3 - 2$. Determine the number of absolute values on $L = \mathbb{Q}(\alpha)$ extending $|.|_p$ on $\mathbb{Q}$, for $p = 2, 3, 5$.

5

(a) Let $\mathbb{Q}_p^{\text{ab}}$ be the maximal abelian extension of $\mathbb{Q}_p$. Define the Weil group $W(\mathbb{Q}_p^{\text{ab}}/\mathbb{Q}_p)$ and show that it is dense in $\text{Gal}(\mathbb{Q}_p^{\text{ab}}/\mathbb{Q}_p)$.

(b) Let ζ_{p^n} be a primitive p^n-th root of unity and $K_n = \mathbb{Q}_p(\zeta_{p^n})$. Show that $K_n/\mathbb{Q}_p$ is a totally ramified Galois extension of degree $p^{n-1}(p-1)$, and that there is an isomorphism

$$\psi_n : \text{Gal}(K_n/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

Explain how this is used to define the Artin map for $\mathbb{Q}_p$.

(c) State the main result of local class field theory (local Artin reciprocity) and use it to show that $N_{K_n/\mathbb{Q}_p}(K_n^\times) = \langle p \rangle \times (1 + p^n\mathbb{Z}_p) \subset \mathbb{Q}_p^\times$.

END OF PAPER