MAMA/133, NST3AS/133, MAAS/133

MAT3

# MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 1:30 pm to 4:30 pm

## PAPER 133

## GEOMETRIC GROUP THEORY

### Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Consider a finite presentation $\langle S \mid R \rangle$ for a group $G$. Define the *area* of a word $w \in F(S)$ that represents the identity in $G$, and the *Dehn function* of the presentation $\langle S \mid R \rangle$.

(b) State the normal form theorem for amalgamated free products. In the case of a free product $A * B$, deduce a sufficient condition for an element

$$g = a_1 b_1 a_2 \dots a_k b_k$$

of $A * B$ to be non-trivial.

Now consider the infinite dihedral group $D_\infty$, given by the presentation $\langle a, b \mid a^2, b^2 \rangle$.

(c) Describe the Bass–Serre tree of the natural decomposition of $D_\infty$ as a free product. Deduce that $D_\infty$ is quasi-isometric to $\mathbb{R}$. [*You may use results from the course, as long as they are clearly stated.*]

(d) Let $w$ be a reduced word in $a, b$ of length $n$ that represents the identity in $D_\infty$. Prove that $n$ is even. Prove furthermore that the area of $w$ is at most $n/2$.

(e) Prove that the Dehn function of the given presentation of $D_\infty$ satisfies $\delta(2n) = n$.

**2** Let $T$ be a tree, and let $\phi$ be a combinatorial isometry of $T$. Recall that either $\phi$ fixes a point of $T$ (i.e. $\phi$ is *elliptic*) or $\phi$ preserves a line $\mathrm{Axis}(\phi)$ in $T$ (i.e. $\phi$ is *hyperbolic*).

(a) Assuming that $\phi$ is elliptic, prove that the set of fixed points $\mathrm{Fix}(\phi)$ is path-connected.

(b) Assuming that $\phi$ is hyperbolic, prove that $\mathrm{Axis}(\phi)$ is the set of points moved a minimal distance by $\phi$. Prove that $\mathrm{Axis}(\phi^n) = \mathrm{Axis}(\phi)$ for all integers $n \neq 0$.

Now consider the Baumslag–Solitar group

$$G = BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle ,$$

for non-zero positive integers $m \neq n$.

(c) Prove that, whenever $G$ acts combinatorially on a tree $T$, $a$ acts elliptically.

(d) Deduce that $G$ does not decompose non-trivially as a free product. [*You may assume without proof that $a$ has infinite order in $G$.*]

**3** Throughout this question a group $G$, generated by a finite set $S$, has a subgroup $H$ generated by a finite set $T$. Let $d_S$ and $d_T$ be the associated word metrics.

(a) We say that $H$ is *quasi-isometrically embedded* in $G$ if there are $\lambda \geqslant 1$ and $\epsilon \geqslant 0$ such that the inclusion map $(H, d_T) \to (G, d_S)$ is a $(\lambda, \epsilon)$-quasi-isometric embedding. Prove that this definition is independent of the choices of $S$ and $T$.

A subspace $Y$ of a geodesic metric space $X$ is called $K$-*quasiconvex* (for $K \geqslant 0$) if any geodesic $[y_1, y_2]$ with endpoints in $Y$ is contained in the closed $K$-neighbourhood of $Y$.

(b) Prove that, if $H$ is $K$-quasiconvex in $\mathrm{Cay}_S(G)$, then $H$ is generated by the set of elements of $H$ of length at most $2K + 1$ with respect to $S$. Deduce that $H$ is quasi-isometrically embedded in $G$.

(c) Give an example of $G$ generated by a finite $S$ and a finitely generated subgroup $H$, such that $H$ is quasi-isometrically embedded in $G$ but $H$ is not $K$-quasiconvex in $\mathrm{Cay}_S(G)$ for any $K$.

(d) Now suppose that $G$ is hyperbolic. Prove that, if $H$ is quasi-isometrically embedded in $G$, then $H$ is quasiconvex in $\mathrm{Cay}_S(G)$.

[*Hint: In part (d), you may use results from the course as long as they are clearly stated.*]

**4** Consider a group $\Gamma$ acting by isometries on a proper metric space $X$. For any $\phi \in \Gamma$, define the *displacement function*, $d_\phi : X \to \mathbb{R}$, by

$$d_\phi(x) = d(x, \phi(x))\,.$$

(a) Prove that $d_\phi$ is continuous.

Now suppose that $\inf_{x \in X} d_\phi(x) = 0$.

(b) Suppose that the action of $\Gamma$ on $X$ is cocompact. Prove that, for any basepoint $x_0 \in X$, there is a constant $C$, a sequence $(\phi_n)$ of conjugates of $\phi$ and a sequence $(x_n)$ of points in $X$ such that $d(x_0, x_n) \leqslant C$ and $d_{\phi_n}(x_n) \to 0$.

(c) Now suppose in addition that the action of $\Gamma$ on $X$ is properly discontinuous. Prove that $\phi$ fixes a point in $X$.

(d) Prove that any Fuchsian group which acts cocompactly on the hyperbolic plane does not contain a non-trivial parabolic isometry. [*You may use without proof that*

$$d(x_1 + iy, x_2 + iy) \leqslant \frac{|x_1 - x_2|}{y}$$

*in the upper half-plane model of the hyperbolic plane.*]

## END OF PAPER