UNIVERSITY OF CAMBRIDGE

MAMA/130, NST3AS/130, MAAS/130

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 1:30 pm to 3:30 pm

PAPER 130

RAMSEY THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove Ramsey's Theorem for r-sets.

(b) Deduce by a compactness argument that for any positive integers r, k and m, there exists a positive integer n such that whenever $[n]^{(r)}$ is k-coloured, we can find a monochromatic set of size m.

(c) (i) Let t be a prime number and k be a positive integer. Show that whenever $\mathbb{N}$ is k-coloured, there exists an infinite sequence $(x_i)_{i\geqslant 1}$ such that the set

$$\Big\{\sum_{j=1}^{t} x_{i_j}^2 : i_1 < \cdots < i_t\Big\}$$

is monochromatic. Show that $(x_i)_{i\geqslant 1}$ can even be taken to be strictly increasing.

(ii) Is the same true if instead we ask for the set

$$\Big\{\sum_{j=1}^{t} x_{i_j}^2 : i_1 < \cdots < i_t\Big\} \cup \Big\{\prod_{j=1}^{t} x_{i_j}^2 : i_1 < \cdots < i_t\Big\}$$

to be monochromatic? The sequence may not necessarily be injective.

2

(a) State and prove the Hales-Jewett Theorem about combinatorial lines. Deduce Van der Waerden's Theorem about arithmetic progressions.

(b) State and prove the strengthened Van der Waerden Theorem.

(c) Consider the grid $\mathbb{N}^2$. We say that three points form a *2-dimensional arithmetic progression* if they are of the form $(a, b), (a + r, b + r), (a + 2r, b + 2r)$ for some positive integers a, b, r. Show directly (i.e. without applying Van der Waerden) that whenever $\mathbb{N}^2$ is finitely coloured, there exists a monochromatic 2-dimensional arithmetic progression.

(d) Deduce from (c) and a product argument that whenever $\mathbb{N}^2$ is 2-coloured we can find a monochromatic set of the form $\{(a, b), (a + r, b + r), (a + 2r, b + 2r), (a + 2r, b)\}$.

3

(a) Let A be a partition regular rational matrix. Show that A has the columns property.

(b) State Rado's Partition Regularity Theorem. Let $A_1, \ldots A_n$ be rational matrices such that, for every i, the equation $A_i x = 0$ has a monochromatic solution whenever $\mathbb{N}$ is finitely coloured. Deduce that for every finite colouring of $\mathbb{N}$ there exists $x_1, \ldots, x_n$ such that $A_i x_i = 0$ for all i, and the set of their entries is monochromatic.

(c) (i) Let $\mathbb{N}$ be finitely coloured. Show that there exists an infinite sequence $(x_i)_{i \geqslant 1}$ such that the set $\{x_i + 2x_j + 3x_k : i < j < k\}$ is monochromatic. Show that $(x_i)_{i \geqslant 1}$ can even be taken to be strictly increasing.

(ii) Does the same hold if we instead ask for the set $\{\sum_{i \in I} x_i : |I| < \infty\} \cup \{x_i + 2x_j + 3x_k : i < j < k\}$ to be monochromatic?

4

(a) Let $X \subset \mathbb{R}^n$ be a finite set that is not spherical. Show that X is not Euclidean Ramsey.

(b) Show that if $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are both finite Euclidean Ramsey sets, then $X \times Y \subset \mathbb{R}^{n+m}$ is Euclidean Ramsey. Deduce that, for any acute-angled triangle T and positive real number t, the triangular prism with base T and height t is Euclidean Ramsey. *[A triangular prism is a three-dimensional shape where the two bases are identical triangles, and the rectangular faces connecting them are perpendicular to the bases.]*

(c) Show directly that the trapezium with side lengths 9, 5, 9, 10, in this order, is Euclidean Ramsey.

(d) Show that, for every $n \geqslant 2$, there exists a 6-colouring of $\mathbb{R}^n$ such that there is no monochromatic copy of a unit equilateral triangle together with its centre.

END OF PAPER