MAMA/129, NST3AS/129, MAAS/129

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 1:30 pm to 3:30 pm

PAPER 129

INTRODUCTION TO ADDITIVE COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Except where a proof is explicitly requested, you may use any theorem from the course without proof provided it is stated clearly.

1 Let $p \geqslant 2$ be a prime and let $n > 1$ be an integer.

(a) State and prove the Freiman-Ruzsa theorem in $\mathbb{F}_p^n$.

(b) Give an example to show that the Freiman-Ruzsa theorem is (close to) best possible. Justify your answer.

(c) Show that there exists a constant $C > 1$ such that if $A \subseteq \mathbb{F}_p^n$ has at least $\eta|A|^3$ additive quadruples for some $\eta > 0$ (i.e. there are at least $\eta|A|^3$ quadruples $(x, y, z, w) \in A^4$ with $x + y = z + w$), then $A + A - A - A$ contains a subspace $V \leqslant \mathbb{F}_p^n$ of size $|V| \geqslant p^{-\eta^{-C}}|A|$.

Comment on this result in relation to Bogolyubov's lemma.

2 Let G be a finite abelian group, and let $\widehat{G}$ denote its character group.

(a) Given $\Gamma \subseteq \widehat{G}$ and $\rho > 0$, define the *Bohr set* $B(\Gamma, \rho)$. State a lower bound for the size of $B(\Gamma, \rho)$.

(b) Let $f : G \to [0, 1]$, let $\delta = \mathbb{E}_{x\in G} f(x) > 0$, and let $\epsilon > 0$. By considering the Fourier expansion for $f * f * f$, show that there exists $\Gamma \subseteq \widehat{G}$ of size $|\Gamma| \leqslant \epsilon^{-2}\delta^{-1}$ such that for all $x \in G$ and all $y \in B(\Gamma, \epsilon)$

$$|f * f * f(x + y) - f * f * f(x)| < 3\epsilon\delta^2.$$

(c) Deduce that if $A \subseteq G$ is a set of density $\alpha = |A|/|G| > 0$, then there exists $\Gamma \subseteq \widehat{G}$ of size $|\Gamma| \leqslant 2\alpha^{-3}$ and $x \in G$ such that $x + B(\Gamma, \alpha) \subseteq A + A + A$.

Let $G = \mathbb{Z}/N\mathbb{Z}$ with $N > 1$ prime.

(d) Show that for any $\Gamma \subseteq \widehat{\mathbb{Z}/N\mathbb{Z}}$ and $\rho > 0$, $B(\Gamma, \rho)$ contains an arithmetic progression of length at least $\frac{1}{8}\rho N^{1/|\Gamma|}$ centred at 0.

(e) Deduce that if $A \subseteq \mathbb{Z}/N\mathbb{Z}$ is a set of density $\alpha = |A|/N > 0$, then $A+A+A$ contains an arithmetic progression of length at least $\frac{1}{8}\alpha N^{\frac{1}{2}\alpha^3}$.

3 Let G be a finite abelian group. For $x \in G$, let τ_x denote the shift-by-x operator. That is, for any function $g : G \to \mathbb{C}$, $\tau_x g(y) = g(y + x)$ for all $y \in G$.

(a) Carefully state Croot and Sisask's almost-periodicity result. Outline a strategy for its proof.

Let $p \geqslant 2$ be a prime and let $n > 1$ be an integer. Suppose that for every $f : \mathbb{F}_p^n \to \mathbb{C}$, every integer $q \geqslant 2$ and every real number $\epsilon > 0$, there are $k \leqslant q/\epsilon^2$ characters $\gamma_1, \gamma_2, \ldots, \gamma_k \in \widehat{\mathbb{F}_p^n}$ and complex numbers $c_1, c_2, \ldots, c_k$ with $|c_i| = 1$ for all $i \in [k]$ such that

$$\Big\| f - \frac{1}{k} \sum_{i=1}^{k} c_i \gamma_i \|\widehat{f}\|_{\ell^1(\widehat{\mathbb{F}_p^n})} \Big\|_{L^q(\mathbb{F}_p^n)} \leqslant \epsilon \|\widehat{f}\|_{\ell^1(\widehat{\mathbb{F}_p^n})}.$$

(b) Show that given $f : \mathbb{F}_p^n \to \mathbb{C}$, $q \geqslant 2$ and $\epsilon > 0$, there is a subspace $W \leqslant \mathbb{F}_p^n$ of codimension at most $4q/\epsilon^2$ such that for all $x \in W$

$$\|\tau_x f - f\|_{L^q(\mathbb{F}_p^n)} \leqslant \epsilon \|\widehat{f}\|_{\ell^1(\widehat{\mathbb{F}_p^n})}.$$

(c) Making suitable choices of f, q and ϵ, show that if $A \subseteq \mathbb{F}_p^n$ is a subset of density $\alpha = |A|/p^n > 0$, then $A + A$ contains a coset of a subspace of dimension at least $\alpha^2 n/(8p^2)$.

[*Hint: You may assume that for any function* $g : \mathbb{F}_p^n \times \mathbb{F}_p^n \to [-1, 1]$ *and any subset* $D \subseteq \mathbb{F}_p^n$,

$$\mathbb{E}_{y \in \mathbb{F}_p^n} \sup_{x \in D} |g(x, y)| \leqslant |D|^{1/q} \sup_{x \in D} \left(\mathbb{E}_{y \in \mathbb{F}_p^n} |g(x, y)|^q \right)^{1/q}.]$$

4 Let $p \geqslant 2$ be a prime and let $n > 1$ be an integer.

(a) Define the U^3 inner product and the U^3 norm for functions $(f_\epsilon : \mathbb{F}_p^n \to \mathbb{C})_{\epsilon \in \{0,1\}^3}$ and $f : \mathbb{F}_p^n \to \mathbb{C}$, respectively. State the Gowers-Cauchy-Schwarz inequality.

(b) (i) Let $A \subseteq \mathbb{F}_p^n$ be a set of density $\alpha = |A|/p^n > 0$. Show that $\|1_A\|_{U^3(\mathbb{F}_p^n)} \geqslant \alpha$.

(ii) Let $q : \mathbb{F}_p^n \to \mathbb{F}_p$ be of the form $q(x) = x^T M x$ for some symmetric $n \times n$ matrix M with entries in $\mathbb{F}_p$, and let $h : \mathbb{F}_p^n \to \mathbb{C}$ be defined by $h(x) = e^{2\pi i q(x)/p}$. Show that $\|h\|_{U^3(\mathbb{F}_p^n)} = 1$.

(iii) Let $f : \mathbb{F}_p^n \to \mathbb{C}$. Show that if $|\langle f, h\rangle_{L^2(\mathbb{F}_p^n)}| \geqslant \delta$ for some $\delta > 0$, then $\|f\|_{U^3(\mathbb{F}_p^n)} \geqslant \delta$.

Let $N > 1$ be an integer.

(c) Let $S \subseteq \mathbb{F}_p^n$ be a subset of density $\sigma = |S|/p^n > 0$, and let $\phi : S \to \mathbb{F}_p^N$ be a function with the property that whenever $x_1, x_2, x_3, x_4 \in S$ satisfy $x_1 + x_2 = x_3 + x_4$, then $\phi(x_1) + \phi(x_2) = \phi(x_3) + \phi(x_4)$. Let $g : \mathbb{F}_p^{n+N} \to \mathbb{C}$ be defined by $g(x, y) = 1_S(x) e^{2\pi i \phi(x)^T y/p}$.

Show that

$$\|g\|_{U^3(\mathbb{F}_p^{n+N})} \geqslant \sigma.$$

State a theorem you could apply in this situation.

END OF PAPER