UNIVERSITY OF CAMBRIDGE

MAMA/128, NST3AS/128, MAAS/128

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 1:30 pm to 3:30 pm

PAPER 128

FORCING AND THE CONTINUUM HYPOTHESIS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the *Lévy Reflection Theorem.*

(b) Let φ be a sentence in the language of set theory and assume that for all finite $T \subseteq \mathsf{ZFC}$ there is a finite $T^* \subseteq \mathsf{ZFC}$ such that if M is a countable transitive model of T^*, then there is a countable transitive model of $N \supseteq M$ of $T + \varphi$. Show that $\mathrm{Con}(\mathsf{ZFC})$ implies $\mathrm{Con}(\mathsf{ZFC} + \varphi)$.

If $R \subseteq \omega \times \omega$, we call R a *code* if (ω, R) is a wellorder, i.e., is isomorphic to a unique countable ordinal α, called the *representation* of R. A level of the constructible hierarchy $\mathbf{L}_\lambda$ is called a *coding level* if

(i) λ is a limit ordinal,

(ii) every code in $\mathbf{L}_\lambda$ has a representation in $\mathbf{L}_\lambda$, and

(iii) every ordinal in $\mathbf{L}_\lambda$ has a code in $\mathbf{L}_\lambda$.

(c) Give a concrete example of a limit ordinal $\lambda > \omega$ such that $\mathbf{L}_\lambda$ is not a coding level. Justify your answer.

(d) Assume $\mathbf{V}$=$\mathbf{L}$. What is the size of the set $\Gamma := \{\lambda < \omega_1\, ; \mathbf{L}_\lambda \text{ is a coding level}\}$? Justify your answer.

[*In this question, you may use results proved in the lectures (in particular, the existence of a condensation sentence σ such that for any transitive set X, $X \models \sigma$ if and only if $X = \mathbf{L}_\lambda$ for a limit ordinal $\lambda > \omega$) as well as standard model-theoretic results (in particular, the Löwenheim-Skolem theorem or the Mostowski collapsing lemma), provided that you state them precisely and correctly.*]

2 In this question, let M be a countable transitive model of set theory, $(\mathbb{P}, \leqslant, \mathbf{1})$ be a partial order in M, and G be $\mathbb{P}$-generic over M.

(a) Briefly explain what *Jerusalem notation* refers to and give the definitions of "$p, q \in \mathbb{P}$ are *incompatible*" and "D is *dense* in $\mathbb{P}$" both in standard and Jerusalem notation.

(b) Show that the axiom schema of Replacement holds in the generic extension $M[G]$. [*You may use the Lévy Reflection Theorem, the Forcing Theorem, and the fact that the axiom schema of Separation holds in $M[G]$ without proof.*]

We write $\omega^{<\omega}$ for the set of finite sequences of natural numbers and ω^{ω} for the set of functions from ω to ω.

For $f, g \in \omega^{\omega}$ we say that g *dominates* f if the set $\{n\,;\, g(n) \leqslant f(n)\}$ is finite. Define $\mathbb{D} := \{(s, f)\,;\, s \in \omega^{<\omega},\, f \in \omega^{\omega}\}$ with $(s, f) \leqslant (t, g)$ if and only if $s \supseteq t$, $f(n) \geqslant g(n)$ for all n, and if $n \in \mathrm{dom}(s) \backslash \mathrm{dom}(t)$, then $s(n) \geqslant g(n)$.

(c) Prove that if G is $\mathbb{D}$-generic over M, then there is some $d \in \omega^{\omega} \cap M[G]$ such that d dominates every function $h \in \omega^{\omega} \cap M$.

(d) Suppose that $M \models \mathsf{CH}$ and G is $\mathbb{D}$-generic over M. Determine the value of $2^{\aleph_0}$ in $M[G]$. Justify your answer. [*You may use any results proved in the lectures, provided that you state them precisely and correctly.*]

END OF PAPER