UNIVERSITY OF CAMBRIDGE

MAMA/126, NST3AS/126, MAAS/126

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 11 June 2025 1:30 pm to 4:30 pm

PAPER 126

ABELIAN VARIETIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $A \to B \twoheadrightarrow C = B/I$ be ring homomorphisms. Show that there is an exact sequence of C-modules

$$I/I^2 \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

giving explicit descriptions of the maps.

(ii) Let K/k be a field extension, $K[t]$ the polynomial algebra. Show that

$$\Omega_{K[t]/k} \simeq \Omega_{K/k} \otimes_K K[t] \oplus K[t].dt\,.$$

Now let $L/K/k$ be field extensions, with L/K finite.

(a) Show that if $L = K(b)$ is a simple extension of K, and $f \in K[t]$ is the minimal polynomial of b over K, then there is an exact sequence of L-vector spaces

$$L \longrightarrow \Omega_{K/k} \otimes_K L \oplus L.dt \longrightarrow \Omega_{L/k} \longrightarrow 0$$

in which the image of $1 \in L$ under the first map is of the form $(*, f'(b)\,dt)$.

(b) Deduce that if L/K is separable, then $\Omega_{K/k} \otimes_K L \xrightarrow{\sim} \Omega_{L/k}$, and that for general L/K (not necessarily simple) one always has $\dim_L \Omega_{L/k} \geqslant \dim_K \Omega_{K/k}$. Give an example for which $\dim_L \Omega_{L/k} > \dim_K \Omega_{K/k}$.

2 (i) Let $f\colon X \to S = \operatorname{Spec} A$ be a proper morphism, with A Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is A-flat.

State a theorem on the existence of a finite complex of finite locally free A-modules computing the cohomology of $\mathcal{F}$. Use it to show that:

(a) if S is connected then the Euler characteristic $\chi(X_s, \mathcal{F}_s)$ is independent of $s \in S$;

(b) for every $p \geqslant 0$ and $n \geqslant 0$, the subset

$$\{s \in S \mid \dim_{\kappa(s)} H^p(X_s, \mathcal{F}_s) \geqslant n\}$$

is closed.

(ii) Compute the Hilbert polynomial $P(X, \mathcal{F}, t)$ in each of the following cases (where k is a field):

(a) $X = \mathbb{P}^2_k$, $\mathcal{F} = \mathcal{O}_X$;

(b) $X \subset \mathbb{P}^2_k$ a smooth curve of degree $m \geqslant 1$, $\mathcal{F} = \mathcal{O}_X(D)$ for D a divisor on X of degree $d \in \mathbb{Z}$.

3

(i) State a version of Mumford's rigidity lemma. Let G be a k-group variety and X an abelian variety over k. Suppose that $f\colon X \to G$ is a morphism of k-schemes such that $f(e_X) = e_G$. Show that f is a homomorphism. Show that if X is only assumed to be a group variety, then f need not be a homomorphism.

(ii) Let G be a k-group scheme, S any k-scheme and $x \in G(S)$. Define the left translation morphism $T_x\colon G \times_k S \to G$, and show that the product morphism $T_{x/S} = (T_x, pr_2)\colon G \times_k S \to G \times_k S$ is an isomorphism.

Show that there are isomorphisms

$$T_x^* \Omega_{G/k} \xrightarrow{\sim} \Omega_{G \times_k S/S} = pr_1^* \Omega_{G/k}$$

and

$$x^* \Omega_{G/k} \xrightarrow{\sim} \mathcal{O}_S \otimes_k \Omega_{G/k}(e)$$

of quasicoherent sheaves on $G \times_k S$ and S, respectively.

Hence show that $\Omega_{G/k}$ is a free $\mathcal{O}_G$-module.

4 In this question, all varieties are over an algebraically closed field k.

(i) State the Theorem of the Square.

(ii) Define the map $\phi_{\mathcal{L}}\colon X(k) \to \operatorname{Pic} X$ attached to a line bundle $\mathcal{L}$ on an abelian variety X, and show that it is a homomorphism.

Show that $\operatorname{Pic}^0 X = \{\mathcal{L} \in \operatorname{Pic} X \mid \phi_{\mathcal{L}} = 0\}$ is a subgroup of $\operatorname{Pic} X$, and that for every $\mathcal{L}$, $\operatorname{im}(\phi_{\mathcal{L}}) \subset \operatorname{Pic}^0 X$.

(iii) Let X_1 and X_2 be abelian varieties, and $X = X_1 \times_k X_2$. Show that the map

$$\tau\colon \operatorname{Pic} X_1 \times \operatorname{Pic} X_2 \to \operatorname{Pic} X, \quad \tau(\mathcal{L}_1, \mathcal{L}_2) = pr_1^* \mathcal{L}_1 \otimes pr_2^* \mathcal{L}_2$$

is an injective homomorphism. Give an example to show that it is not in general surjective.

(iv) Show that τ induces an isomorphism

$$\operatorname{Pic}^0 X_1 \times \operatorname{Pic}^0 X_2 \xrightarrow{\sim} \operatorname{Pic}^0 X\,.$$

[You may use without proof the fact that if $\mathcal{L}$ is ample then $\phi_{\mathcal{L}}(X(k)) = \operatorname{Pic}^0(X)$.]

END OF PAPER