UNIVERSITY OF CAMBRIDGE

MAMA/120, NST3AS/120, MAAS/120

MAT3
MATHEMATICAL TRIPOS Part III

Thursday 5 June 2025 1:30 pm to 4:30 pm

PAPER 120

LOGIC AND COMPUTABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define *Heyting algebra*.

(b) Show that every finite distributive lattice L is a Heyting algebra.

(c) What does the typed λ-term $\lambda p\colon \phi \times \psi.\ \lambda f : \phi \to (\psi \to 0).\ ((f\,\pi_1(p))\ \pi_2(p))$ correspond to under the Curry-Howard correspondence?

(d) Can $\wedge$ be defined in terms of $\to$ and $\perp$ in intuitionistic propositional logic? Justify your claim.

(e) Show that $\neg(\phi \to \neg\psi) \to (\phi \wedge \psi)$ is not intuitionistically valid.

[*You may assume any results from the lectures that you accurately state without proof.*]

(f) We say that a world w in a Kripke model W *determines* p if $w \Vdash p$ or $w \Vdash \neg p$. Show that if w determines all the primitive propositions within a proposition ϕ and $w \leqslant w'$, then $w \Vdash \phi$ precisely when $w' \Vdash \phi$.

2

(a) State the *Church-Rosser Theorem* for the untyped λ-calculus.

(b) State and prove the *Weak Normalisation Theorem* for the implicational fragment of the simply typed λ-calculus $\lambda(\to)$.

(c) Define what it means for a λ-term F to λ-*define* a function $f\colon \mathbb{N}^k \to \mathbb{N}$.

(d) Suppose λ-terms R and S define the same function. Must R and S be β-equivalent? Justify your answer.

(e) Show that there is a λ-term δ such that $L \equiv_{\beta\eta} (\delta\ L)$ precisely when L is a fixed-point combinator.

3

(a) Define what is meant by Σ_1 and Π_1 formulae in the language of PA.

[*You need not define Δ_0-formulae.*]

(b) Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be a total function and T be a theory in the language of PA that extends PA^-. Explain what it means to say that "f is Σ_1-represented in T".

(c) State and prove the Diagonalisation Lemma for theories in the language of PA.

[*You may assume that $\mathcal{L}_{PA}$ is recursive and that all total recursive functions are Σ_1-represented in PA^-.*]

(d) Fix a Gödel numbering of the $\mathcal{L}_{\mathrm{PA}}$-formulae and let $M \models \mathrm{PA}^-$. Show that there is no $\mathcal{L}_{\mathrm{PA}}$-formula $\theta(x)$ such that, for all $n \in \mathbb{N}$, $M \models \theta(n)$ precisely when n is the Gödel numbering of an $\mathcal{L}_{\mathrm{PA}}$-sentence σ with $M \models \sigma$.

(e) State Tennenbaum's Theorem.

(f) Suppose that a non-recursive set X is canonically coded in a countable model M of PA by some element $c \in M$. Show that the multiplication operation in M cannot be recursive.

[*For this item, you may use any results from the lectures or theorems of PA without proof.*]

END OF PAPER