MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 9:00 am to 12:00 pm

PAPER 119

CATEGORY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the Yoneda Lemma, and deduce that if $\mathcal{C}$ is a small category then representable functors are projective in $[\mathcal{C}, \mathbf{Set}]$. [You may assume that epimorphisms in this category are pointwise surjections.]

State the Special Adjoint Functor Theorem, explaining the terms involved. Deduce that if $\mathcal{C}$ and $\mathcal{D}$ are small categories, then any functor $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ which preserves all small colimits has a right adjoint. In particular, deduce that $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed for any small $\mathcal{C}$.

An object A of a cartesian closed category $\mathcal{E}$ is said to be *tiny* if the functor $(-)^A \colon \mathcal{E} \to \mathcal{E}$ has a right adjoint. If $\mathcal{C}$ is a small category with binary products, show that the representable functors $\mathcal{C}(-, A)$ are tiny in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Conversely, if $\mathcal{C}$ has a terminal object, show that any tiny object F of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is an indecomposable projective (that is, the functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](F, -)$ preserves coproducts and epimorphisms).

2 Define the *unit* and *counit* of an adjunction $(F \dashv G)$. For such an adjunction, show that the following three conditions are equivalent:

(i) G is full and faithful.

(ii) The counit of the adjunction is an isomorphism.

(iii) There is an isomorphism between FG and the identity functor.

Deduce that if F has both a left adjoint L and a right adjoint R, then L is full and faithful if and only if R is.

Now consider a double adjunction $(L \dashv F \dashv R)$ where F is full and faithful; let η and ϵ denote the unit and counit of $(L \dashv F)$, and α and β those of $(F \dashv R)$. Show that the composites

$$R \xrightarrow{R\eta} RFL \xrightarrow{(\alpha_L)^{-1}} L \qquad \text{and} \qquad R \xrightarrow{(\epsilon_R)^{-1}} LFR \xrightarrow{L\beta} L$$

are equal [*hint*: consider their images under F]. Show also that this natural transformation $\theta \colon R \to L$ is pointwise monic if and only if L acts faithfully on morphisms whose domains are in the image of F. [Given a morphism $f \colon B \to RA$, consider $F(\theta_A f)$.]

3 Explain what is meant by a *limit* of a diagram $D\colon J \to \mathcal{C}$, and show that all finite limits may be constructed from finite products and equalizers.

Let **Met** denote the category of metric spaces and non-expansive maps. Show that **Met** has binary products. Show also that any parallel pair $1 \rightrightarrows X$ in **Met**, where 1 denotes the one-point space, has a coequalizer which is preserved by the forgetful functor $\mathbf{Met} \to \mathbf{Set}$.

Now let $X = \{0, 1, 5, 6\}$ and $Y = \{0, 3\}$, both equipped with the Euclidean metric (i.e. $d(x, y) = |x - y|$). Show that the coequalizer of $(0, 6)\colon 1 \rightrightarrows X$ in **Met** is not preserved by the functor $(-) \times Y\colon \mathbf{Met} \to \mathbf{Met}$, and deduce that **Met** is not cartesian closed.

4 Explain briefly what is meant by the *monad* induced by an adjunction, and by an (Eilenberg–Moore) *algebra* for a monad. Given an adjunction $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ inducing a monad $\mathbb{T}$ on $\mathcal{C}$, define the Eilenberg–Moore *comparison functor* $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$, and show that if $\mathcal{D}$ has coequalizers of reflexive pairs then K has a left adjoint. Explain what is meant by the *monadic length* of such an adjunction $(F \dashv G)$.

Let **2** denote the ordered set $\{0, 1\}$ with $0 < 1$, and let $\mathbf{Mono}(\mathbf{Set})$ denote the full subcategory of $[\mathbf{2}, \mathbf{Set}]$ whose objects are injections $(A_0 \rightarrowtail A_1)$. Show that the forgetful functor $\mathbf{Mono}(\mathbf{Set}) \to \mathbf{Set} \times \mathbf{Set}$ has a left adjoint, and determine the monadic length of this adjunction. [You may assume the result that reflections are monadic.]

5 Explain carefully what is meant by the statement that limits of shape I commute with colimits of shape J in a category $\mathcal{C}$.

Define the notions of *filtered* and *weakly filtered* category. Assuming the result that filtered colimits commute with all finite limits in **Set**, prove that weakly filtered colimits commute with connected finite limits in **Set**.

Show also that filtered colimits commute with finite limits in the category **AbGp** of abelian groups. Do cofiltered limits commute with finite colimits in **AbGp**? Justify your answer. [You may find it helpful to consider a suitable pair of inverse sequences in **AbGp**.]

6 Explain what is meant by a *regular category*, and by the category $\mathbf{Rel}(\mathcal{C})$ of relations in a regular category $\mathcal{C}$. [You need not verify that $\mathbf{Rel}(\mathcal{C})$ is indeed a category.] Show that a morphism $f : A \nrightarrow B$ of $\mathbf{Rel}(\mathcal{C})$ is a left adjoint (that is, there exists $g : B \nrightarrow A$ with $1_A \leqslant gf$ and $fg \leqslant 1_B$) if and only if it is the graph of a morphism $A \to B$ in $\mathcal{C}$.

Now let L be a frame (that is, a complete lattice satisfying the distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$ for all $a \in L$ and $S \subseteq L$). The category $\mathbf{Mat}(L)$ of 'L-valued matrices' has all sets as objects, and morphisms $f : A \nrightarrow B$ are functions $f : A \times B \to L$, the composite of f and $g : B \nrightarrow C$ being the function $((a, c) \mapsto \bigvee\{f(a, b) \wedge g(b, c) \mid b \in B\})$. [Again, you need not verify that this composition is associative.] The morphisms $A \nrightarrow B$ in $\mathbf{Mat}(L)$ are partially ordered pointwise (i.e. $f \leqslant g$ if and only if $f(a, b) \leqslant g(a, b)$ for all a and b). Show that $f : A \nrightarrow B$ is a left adjoint in $\mathbf{Mat}(L)$ if and only if $\bigvee\{f(a, b) \mid b \in B\} = 1$ for all $a \in A$ and $f(a, b) \wedge f(a, b') = 0$ whenever $b \neq b'$. Deduce that if $L = \Omega(X)$ is the open-set lattice of a connected topological space X, then the left adjoints in $\mathbf{Mat}(L)$ form a category isomorphic to $\mathbf{Set}$.

END OF PAPER