UNIVERSITY OF CAMBRIDGE

MAMA/118, NST3AS/118, MAAS/118

MAT3
MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 9:00 am to 12:00 pm

PAPER 118

COMPLEX MANIFOLDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that $\pi : E \to X$ is a holomorphic line bundle over a complex manifold X. State precisely what it means for E to have a *Hermitian inner product*.

Suppose such a Hermitian inner product on E is given. State the definition of a *Chern connection* on E with respect to this inner product. Give an expression for the Chern connection in terms of this Hermitian inner product. [There is no need to prove that this gives the unique Chern connection, but you should show that your formula is well-defined].

Consider the line bundle $E = \mathcal{O}(-1)$ over $\mathbb{CP}^n$. Give an expression for a Hermitian inner product on E and write down the connection 1-forms for the Chern connection on E in holomorphic local coordinates on $\mathbb{CP}^n$. Use this to find a closed 2-form representing the first Chern class of E.

2 Suppose Y is a closed (smooth) submanifold of a compact complex manifold X, and let J be the almost-complex structure on TX. Prove that if the tangent space $T_pY \subset T_pX$ is a J_p-invariant subspace for every $p \in Y$, then Y is a *complex submanifold* of X.

Now suppose Y is a complex submanifold of X. Define the *holomorphic normal bundle* of Y. State precisely the *adjunction formula* for a smooth hypersurface Y in X and give a proof.

Now suppose $X = \mathbb{CP}^n \times \mathbb{CP}^m$ and $Y \subset X$ is given by the vanishing locus of a bihomogeneous polynomial F of bidegree (d_1, d_2). Give criteria on F and d_1, d_2 under which Y will be a complex submanifold with trivial canonical bundle K_Y.

3 Suppose X is a compact connected complex manifold of dimension n and $p \in X$ is a point. Define the *blowup* $\sigma : \tilde{X} \to X$ of X at the point p, and the *exceptional divisor* E in $\tilde{X}$.

If $Y \subset X$ is a smooth hypersurface, show that the *proper transform* $\tilde{Y}$ of Y, defined as the closure of $\sigma^{-1}(Y \setminus \{p\})$ in $\tilde{X}$, is a smooth hypersurface in $\tilde{X}$.

Define the holomorphic line bundle $\mathcal{O}(E)$ on $\tilde{X}$ associated to the divisor E. Show that $\sigma^*\mathcal{O}(Y) \cong \mathcal{O}(\tilde{Y} + mE)$ as holomorphic line bundles and determine $m \in \mathbb{Z}$.

Show that there is a bijection between the set of holomorphic sections of $\mathcal{O}(E)$, and the set of meromorphic functions f on $\tilde{X}$ such that $(f) + E \geqslant 0$.

4 Suppose X is a compact complex manifold with a Riemannian metric g. State what it means for g to be a *Kähler metric* on X.

Now suppose g is a Kähler metric on X. Define the *Lefschetz operator* $L : C^\infty(X, \wedge^{p,q}T^*X) \to C^\infty(X, \wedge^{p+1,q+1}T^*X)$ and its adjoint Λ. Show the Kähler identities $[L, \partial] = 0$ and $[L, \bar{\partial}] = 0$.

Define the *Laplacians* Δ, Δ_∂ and $\Delta_{\bar{\partial}}$ and show that $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$. [You may assume the Kähler identity $[\Lambda, \partial] = i\bar{\partial}^*$].

Use this to show that Λ^k maps ∂-harmonic (p, q)-forms to ∂-harmonic $(p-k, q-k)$-forms (for $0 \leqslant k \leqslant p, q \leqslant n$). [You may use additional Kähler identities if you provide a proof].

END OF PAPER