UNIVERSITY OF CAMBRIDGE

MAMA/115, NST3AS/115, MAAS/115

MAT3

MATHEMATICAL TRIPOS Part III

Friday 6 June 2025 1:30 pm to 4:30 pm

PAPER 115

DIFFERENTAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let M be a manifold. What is meant by an *embedding* of a manifold N in M?

Explain, stating accurately any auxiliary results you require from the course, why the inclusion map of S^n as the unit sphere in $\mathbb{R}^{n+1}$ is an embedding.

Construct an embedding of the product of spheres $S^{n_1} \times \ldots \times S^{n_k}$ in $\mathbb{R}^{n+1}$, where $n = \sum_{i=1}^{k} n_i$ and $n_i \geqslant 1$ for all i.

(b) Define the *de Rham cohomology* $H^p(M)$ of a manifold M. State the Poincaré lemma.

Assume that M is connected and let $\varphi : U \to \mathbb{R}^n$ be a coordinate chart defined on an open subset $U \subset M$ and let $B \subset U$ be such that $\varphi(B)$ is a closed ball in $\mathbb{R}^n$. For $n > 1$, show that if $H^1(M \setminus B) = 0$ then $H^1(M) = 0$. Does the result hold when $n = 1$? Justify your answer.

By considering an appropriate nowhere-vanishing differential n-form on S^n, or otherwise, show that $H^{n+1}(S^n \times I) = 0$ for all n, where $I \subset \mathbb{R}$ is a non-empty finite open interval.

[*You may* not *use the de Rham theorem or any results from algebraic topology concerning (co)homology groups of topological spaces.*]

2 (a) What is the *wedge product* of differential forms on a manifold?

Show that each $\eta \in \Lambda^{n-1}((\mathbb{R}^n)^*)$ can be written as $\xi \wedge \eta_0$ for some $\xi \in (\mathbb{R}^n)^*$ and $\eta_0 \in \Lambda^{n-2}((\mathbb{R}^n)^*)$, where $n \geqslant 3$.

Let M be a compact oriented manifold. Define the *integral* $\int \omega$ of a differential n-form ω over M, where $n = \dim M$. For a smooth map of manifolds $\varphi : M \to N$ define the induced *pull-back* map φ^* for the differential forms. Let ω be nowhere-zero and let $F : M \to M$ be a diffeomorphism such that $F^*\omega = \omega$. Show that

$$\int_M (h \circ F)\, \omega = \int_M h\, \omega$$

for all $h \in C^\infty(M)$.

[*You may assume the existence of a partition of unity if you accurately state this result.*]

(b) Let now M be an oriented Riemannian manifold with a Riemannian metric g. Explain what is meant by the inner product induced by g on the bundles $\Lambda^p T^* M$ of differential p-forms. Define the *volume form* ω_g of g and the *Hodge* $*$*-operator*, showing that $*$ is well-defined.

Deduce from Stokes' theorem the expression for the formal L^2 adjoint δ of the exterior derivative d in terms of $*$ and d. Define the *Laplace–Beltrami operator* Δ for the differential forms on M.

Let $f \in C^\infty(M)$ and let $*_f$ and Δ_f denote the Hodge $*$-operator the Laplace–Beltrami operator obtained using on M the Riemannian metric $e^{2f} g$ in place of g. For a differential form $\alpha \in \Omega^p(M)$ determine $*_f(\alpha)$ in terms of $*\alpha$ and f.

If f is constant, show that $\Delta_f \alpha = e^{-2f} \Delta\alpha$ for each differential form α on M.

3 (a) Let $E \to B$ be a vector bundle over a manifold B. Show that $E \otimes \Lambda^r T^* B$ for $r = 1, 2, \dots$ is a well-defined vector bundle over B, by constructing an appropriate family of local trivializations and transition functions.

Define a *connection* on E and a *covariant derivative* for the sections of E. Show that every connection A on E induces a well-defined covariant derivative d_A. Show, using an appropriate version of the Leibniz rule, that the d_A extends to a linear map (which we still denote by d_A) on sections of $E \otimes \Lambda^k T^* B$. Give an explicit formulas of the latter d_A in local trivializations.

Define the *curvature form* $F(A)$ of a connection A and determine the expression of $F(A)$ in terms of A in a local trivialization. State and prove the Bianchi identity for $F(A)$.

(b) For a covariant derivative ∇ on E and a vector field X on B we write ∇_X for a composition of ∇ and the contraction of sections of $E \otimes T^* B$ with X. Now let ∇ be a covariant derivative on B (i.e. on the tangent bundle TB). Show that ∇ induces a covariant derivative, still denoted by ∇, on $T^* B$ such that

$$X\langle \alpha, Y \rangle = \langle \nabla_X \alpha, Y \rangle + \langle \alpha, \nabla_X Y \rangle$$

for all vector fields X, Y on B and all differential 1-forms α on B. Determine the relation between the local coefficients of ∇ on $T^* B$ and on TB.

Define what it means for ∇ to be *symmetric*. Show that if ∇ is symmetric, then $(\mathrm{Alt} \circ \nabla)\alpha = d\alpha$ for all $\alpha \in \Omega^1(B)$ where Alt denotes the projection from bilinear forms on $T_p B$ to anti-symmetric bilinear forms on $T_p B$ (with the kernel being the symmetric bilinear forms), for all $p \in B$.

4 (a) If $\pi : E \to M$ is a vector bundle endowed with a connection A and $\gamma : [0,1] \to M$ is a smooth curve, define the *horizontal lift* of γ from a point $p \in E$ with $\pi(p) = \gamma(0)$. Show that the horizontal lift is determined in a local trivialization of E as a solution of a linear ordinary differential equation.

Define a *geodesic* on a Riemannian manifold (M, g). Define the map $\exp_p$ and show that its inverse defines local coordinates around $p \in M$. [You need not check that $\exp_p$ is a smooth map.] Define a *geodesic sphere* centred at p.

State and prove Gauss' lemma.

[*You may assume without proof that the length of $\dot\gamma(t)$ is constant for any geodesic $\gamma(t)$.*]

(b) Show that there exists $\varepsilon > 0$ such that for $X \in T_p M$ with $|X|_g < \varepsilon$ the curve $\gamma(t) = \exp_p(tX)$, $0 \leqslant t \leqslant 1$, is a geodesic and any smooth curve $\sigma : [0,1] \to M$ with $\sigma(0) = p$, $\sigma(1) = \gamma(1)$ has $\mathrm{length}(\sigma) \geqslant \mathrm{length}(\gamma)$. Here $\mathrm{length}(\sigma) = \int_0^1 |\dot\sigma(t)|_g dt$.

Assume now that $\dim M \geqslant 3$. let $X \subset M$ be an embedded submanifold with $\dim X = \dim M - 1$ and let Σ be a geodesic sphere centred at p such that $X \cap \Sigma$ consists of one point $q \neq p$. Show that $T_q X = T_q \Sigma$ as subspaces of $T_q M$.

If $\dim M = 2$, must $T_q X = T_q \Sigma$ hold true? Justify your answer.

END OF PAPER