UNIVERSITY OF CAMBRIDGE

MAMA/114, NST3AS/114, MAAS/114

MAT3
MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 1:30 pm to 4:30 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be the space $S^2/\sim$ where $\sim$ is the smallest equivalence relation on the standard 2-sphere $S^2 \subset \mathbb{R}^3$ such that $(a,b,0) \sim (-a,-b,0)$ for all $(a,b) \in S^1$.

(a) Define the *cellular chain complex* of a cell complex, showing carefully that it is indeed a chain complex. State a lemma that expresses the differential in terms of degrees of maps between spheres.

(b) Give a cell structure on X and use it to compute the cellular homology of X.

(c) Compute the local homology $H_*(X, X \setminus \{x\})$ for $x = [(1,0,0)]$, carefully stating all the theorems you use.

(d) Is X homotopy equivalent to a finite cell complex Y that has exactly one 2-cell? Justify your answer.

2

(a) Describe $H^*(\mathbb{CP}^n;\mathbb{Z})$ as a graded ring for every positive integer n. Briefly justify your answer.

(b) State the *Künneth theorem* and use it to compute $H^*(\mathbb{CP}^2 \times \mathbb{CP}^2;\mathbb{Z})$ as a ring.

Let $X = (\mathbb{CP}^2 \times \mathbb{CP}^2)/A$ be the quotient space obtained by collapsing the subspace

$$A = \{(z,w) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid z = [1:0:0] \text{ or } w = [1:0:0]\}.$$

(c) Compute $H^*(X;\mathbb{Z})$ as a ring.

(d) Show that there are no maps $f\colon S^4 \to X$ and $g\colon X \to S^4$ such that $g \circ f$ is homotopic to id_{S^4}.

3

(a) Define the *compactly supported cohomology* $H_c^*(X)$ of a space X, justifying why it is well-defined. Show that every proper map $f\colon X \to Y$ induces a well-defined map $f^*\colon H_c^*(Y) \to H_c^*(X)$.

[Recall that a map $f\colon X \to Y$ is called proper if for every compact subspace $K \subset Y$ the preimage $f^{-1}(K)$ is also compact.]

(b) State a description of $H_c^*(X)$ as a colimit of relative cohomology groups and use it to show that there is an isomorphism $H_c^*(\mathbb{R}^n) \cong H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ for all positive integers n.

Fix a positive integer n and let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be a proper map satisfying $f(\lambda v) = \lambda f(v)$ for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$.

(c) Show that f induces a well-defined map $\overline{f}\colon \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}$ and an isomorphism of complex vector bundles $\overline{f}^*\gamma_{1,n}^{\mathbb{C}} \cong \gamma_{1,n}^{\mathbb{C}}$.

(d) Show that $f^*\colon H_c^*(\mathbb{C}^n) \to H_c^*(\mathbb{C}^n)$ is the identity. [You may use that complex vector bundles are $\mathbb{Z}$-oriented and isomorphisms of complex vector bundles respect these orientations.]

4 Let d be a positive integer and M be a compact, oriented d-manifold.

(a) Define the *cap product*. [You do not have to check that it is well-defined.] What does it mean to say that a homology class $[M] \in H_d(M)$ is a *fundamental class*? Assuming that such a fundamental class is given, state the *Poincaré duality theorem* for M and describe the map involved.

Suppose that $M = U \cup V$ for two contractible open subsets $U, V \subset M$.

(b) State the small simplex theorem for this open cover.

(c) Show that $[M] \frown \alpha = 0$ for all $\alpha \in H^k(M;\mathbb{Z})$ where $0 < k < d$. Deduce that

$$H_i(M) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, d, \\ 0 & \text{otherwise.} \end{cases}$$

END OF PAPER