MAMA/113, NST3AS/113, MAAS/113

MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a topological space and $j : U \hookrightarrow X$ the inclusion of an open subset. Let $\mathcal{F}$ be a sheaf of abelian groups on U. Define the *extension by zero* of $\mathcal{F}$ to be the sheaf $j_!\mathcal{F}$ associated to the presheaf $j_p\mathcal{F}$ on X defined by

$$(j_p\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Give an example to show that $j_p\mathcal{F}$ need not be a sheaf. Show that $j_!\mathcal{F}$ has stalks given by

$$(j_!\mathcal{F})_P = \begin{cases} \mathcal{F}_P & \text{if } P \in U; \\ 0 & \text{otherwise.} \end{cases}$$

(b) For sheaves of abelian groups $\mathcal{F}$ on U and $\mathcal{G}$ on X, construct a natural morphism

$$j_!j^{-1}\mathcal{G} \to \mathcal{G}.$$

Show that there is a natural isomorphism

$$\mathcal{F} \xrightarrow{\cong} j^{-1}j_!\mathcal{F}.$$

[Note: You may use freely that since j is an inclusion of an open set, $j^{-1}\mathcal{G}$ agrees with the restriction $\mathcal{G}|_U$.]

(c) Let $Z = X \setminus U$ and $i : Z \hookrightarrow X$ be the inclusion. Let $\mathcal{G}$ be a sheaf of abelian groups on X. Show that there is an exact sequence of sheaves

$$0 \to j_!j^{-1}\mathcal{G} \to \mathcal{G} \to i_*i^{-1}\mathcal{G} \to 0.$$

(d) Suppose further that X is a scheme, $U \subseteq X$ an open subscheme. Show that $j_!\mathcal{O}_U$ need not be a quasi-coherent sheaf on X.

2 Let $S = \bigoplus_{n \geqslant 0} S_n$ be a graded ring, $S_+ = \bigoplus_{n>0} S_n$.

(a) Define the projective scheme $\operatorname{Proj} S$.

(b) Define what it means for a morphism of schemes $i : Z \to X$ to be a closed immersion.

Show that if $I \subseteq S$ is a homogeneous ideal of S with $I \subseteq S_+$, then $\operatorname{Proj} S/I$ comes with a natural closed immersion into $\operatorname{Proj} S$. [Feel free to quote results from lecture or the example sheets for this question.]

(c) Suppose $I \subseteq S$ is a homogeneous ideal with $I \subseteq S_+$, $I = \bigoplus_{n \geqslant 1} I_n$ with $I_n = I \cap S_n$. Define

$$I' = \bigoplus_{n \geqslant n_0} I_n$$

for some non-negative integer n_0. Show that $\operatorname{Proj} S/I$ and $\operatorname{Proj} S/I'$ define the same closed subscheme of $\operatorname{Proj} S$.

3 (a) Let X be a Noetherian integral separated scheme which is regular in codimension one. Define a *prime (Weil) divisor* on X. Given a prime divisor Y, explain how to define a group homomorphism

$$\nu_Y : K(X)^* \to \mathbb{Z}$$

capturing the notion of order of vanishing along Y.

(b) Let X be a Noetherian integral separated scheme which is regular in codimension one. Let $Z \subseteq X$ be a closed subset which is the union of prime divisors $Z_1, \ldots, Z_n$. Let $U = X \setminus Z$. Show that there is an exact sequence

$$\mathbb{Z}^n \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0.$$

(c) Now let k be an algebraically closed field and let

$$X = \operatorname{Proj}\bigl(k[x_0, x_1, x_2, x_3]/(x_3^3 - x_0 x_1 x_2)\bigr).$$

Let $L_i = V(x_i, x_3) \subseteq X$, $i = 0, 1, 2$. You may take as given without proof that:
(1) X is a Noetherian integral separated scheme which is regular in codimension one, and
(2) $L_i \subseteq X$ is a prime divisor.

Let $f = x_0/x_3 \in K(X)$. Calculate $\nu_{L_i}(f)$ for $i = 0, 1, 2$.

Calculate $\operatorname{Cl}(X)$.

4 (a) Let X be a topological space, $\mathcal{U}$ an open covering of X, and $\mathcal{F}$ a sheaf on X. Define the *Čech cohomology of $\mathcal{F}$ with respect to the cover $\mathcal{U}$*, $\check{H}^p(\mathcal{U}, \mathcal{F})$.

(b) Let k be a field. Show that $\mathbb{P}^n_k$ does not have a cover by n affine open subsets.

(c) Let X be a scheme with an open cover $\mathcal{U}$. Denote by $\operatorname{Pic}(\mathcal{U})$ the group of line bundles on X up to isomorphism for which $\mathcal{U}$ is a trivializing cover. Show that $\operatorname{Pic}(\mathcal{U}) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf of invertible sections of $\mathcal{O}_X$.

(d) Let X be an integral scheme. Define what it means to give a Cartier divisor on X. What does it mean for two Cartier divisors to be linearly equivalent? Define the Cartier class group $\operatorname{CaCl} X$ of X.

Recall a sheaf $\mathcal{F}$ is *flasque* if all restriction homomorphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective. Using the fact that if $\mathcal{F}$ is flasque, $H^i(X, \mathcal{F}) = 0$ for $i > 0$, show the isomorphism

$$\operatorname{CaCl} X \cong H^1(X, \mathcal{O}_X^*).$$

END OF PAPER