UNIVERSITY OF CAMBRIDGE

MAMA/111, NST3AS/111, MAAS/111

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 17 June 2025 9:00 am to 11:00 am

PAPER 111

COXETER GROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question, let $(V, (-,-))$ be a finite dimensional, real inner product space.

(a) Define a *root system* in V. Given a root system Φ, define a *fundamental system* $\Delta \subseteq \Phi$ and the *positive system* $\Pi \subseteq \Phi$ associated to Δ.

(b) Define the *reflection group* $W(\Phi)$ of a root system Φ. Prove that $W(\Phi)$ acts on the set of fundamental systems in Φ.

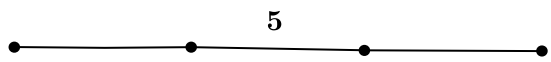
(c) Prove that for any root system $\Phi \subseteq V$, and every fundamental system $\Delta \subseteq \Phi$ with positive system Π, if $w \in W(\Phi)$ and $w \cdot \Delta \subseteq \Pi$, then $w = 1$. Conclude that the stabiliser in $W(\Phi)$ of any fundamental system is trivial.

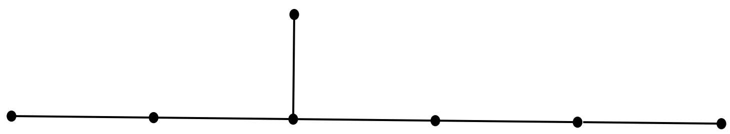
(d) If W is a Coxeter group, show that W contains an element w_0 of maximal length if and only if it is finite. By realising W as a finite reflection group, use part (c) to prove that this element is unique.

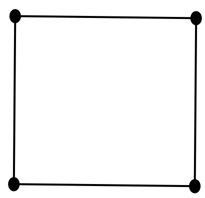
(*In parts (c) and (d) you may assume any results in the course, provided they are stated clearly.*)

2

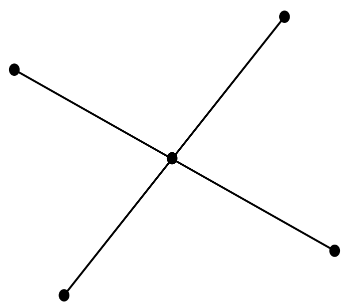
(a) For each of the Coxeter graphs below, state whether the associated bilinear form is *i.* positive definite, *ii.* positive semidefinite, *iii.* non-degenerate. Justify your answer.

1. 

2. 

3. 

(b) State the *exchange condition* and *Matsumoto's theorem* on reduced words in a Coxeter group, and use these results to prove that the Coxeter group with graph given below is not finite.



(c) Alternatively, use the associated bilinear form to prove that the group defined by the graph in part (b) is not finite.

(*In all parts, you may assume any results from the course, provided they are clearly stated.*)

3

Let (W, I, M) be a Coxeter system, with generators $\{x_i, i \in I\}$, $|I| < \infty$. Let (V, σ) be the geometric representation of W, where V has basis $\{e_i : i \in I\}$.

Let (V^*, σ^*) be the dual representation, and for each $i \in I$, set $H_i := \{f \in V^* : f(e_i) = 0\}$, $A_i := \{f \in V^* : f(e_i) > 0\}$ and $A_{-i} := \{f \in V^* : f(e_i) < 0\}$. Set $C := \bigcap_{i \in I} A_i$.

(a) Describe the natural topology on V^* with respect to which H_i is a closed set and A_i, A_{-i}, C are open sets. Write down their closures, and prove that $\sigma^*(w)$ is a homeomorphism for all $w \in W$.

(b) Show that for all $i \in I$, if $\ell(x_i w) > \ell(w)$ then $\sigma^*(w)(C) \subseteq A_i$, and if $\ell(x_i w) < \ell(w)$ then $\sigma^*(w)(C) \subseteq A_{-i}$. Deduce that σ^* is a faithful representation of W.

Define $D := \{f \in V^* : f(e_i) \geqslant 0 \text{ for all } i \in I\}$, and let

$$\mathcal{C} := \bigcup_{w \in W} \sigma^*(w)(C), \quad U := \bigcup_{w \in W} \sigma^*(w)(D) \subseteq V^*$$

It is a fact that D is a fundamental domain for the action of W on U.

(c) Show that the union $\mathcal{C} = \bigcup_{w \in W} \sigma^*(w)(C)$ is disjoint, and when W is finite show that $\mathcal{C}$ can be realised as a Coxeter complex.

(d) Prove that W is finite if and only if $U = V^*$.

(*In part (d), you may assume that if W is infinite and $w \in W$, then the length of w can always be increased by multiplying with x_i on either side for some i. In all parts, you may use any results from the course, provided they are stated clearly.*)

4

Let W be a group, I a set, and let $S := \{s_i : i \in I\} \subseteq W$ be a set of generators for W such that $s_i^2 = 1$ for all $i \in I$. We define the *length* of an element $w \in W$, denoted $\ell_S(w)$ to be the minimal integer n such that $w = s_{i_1} \cdots s_{i_n}$ for some $i_j \in I$.

(a) Prove that for all $w \in W$, $i \in I$, $\ell_S(w^{-1}) = \ell_S(w)$ and $\ell_S(ws_i) = \ell_S(w) - 1, \ell_S(w)$ or $\ell_S(w) + 1$.

We say that W satisfies the *folding condition* if for all $w \in W$, $i, j \in I$, $\ell(s_i w) \neq \ell(w)$ and if $\ell(s_i w) = \ell(w s_j) = \ell(w) + 1$, then either $\ell(s_i w s_j) = \ell(w) + 2$ or $w = s_i w s_j$.

(b) Prove that if W satisfies the folding condition, then given $w = s_{i_1} \cdots s_{i_n} \in W$, $s \in S$, if $\ell(ws) < \ell(w) = n$ then $ws = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$ for some unique $j \in \{1, \ldots n\}$.

For $i, j \in I$, let $m_{i,j}$ be the order of $s_i s_j \in W$.

(c) Define a *braid relation* in W in terms of the numbers $m_{i,j}$, and state what it means for two reduced expressions for an element $w \in W$ to be *braid equivalent*.

(d) Prove that W is a Coxeter group with generators S if and only if it satisfies the folding condition.

(*In part (d), you may assume that if W satisfies the folding condition, then any two reduced expressions for an element of W are braid equivalent. In all parts, you may assume any results from the course, provided they are stated clearly.*)

END OF PAPER